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ABSTRACT

This is unit eleven of a fifteen-unit SMSG secondary school text for high school students. The text is devoted almost entirely to mathematical concepts which all citizens should know in order to function satisfactorily in our society. Chapter topics include coordinate geometry and problem analysis. (MP)

SCHOOL MATHEMATICS STUDY GROUP

UNIT NUMBER ELEVEN

Chapter 17. Coordinate Geometry

Chapter 18. Problem Analysis

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Chapter 17

COORDINATE GEOMETRY

17-1. Slope of a Line

In this section we shall take another look at the line as a subset of the coordinate plane, with special consideration of its direction. We have seen that every pair of distinct points determines a line. Now we shall consider another way of locating a line — by knowing one point on the line and the direction of the line.

Exercises 17-1a

(Class Discussion)

1. On a sheet of graph paper, draw one set of coordinate axes. Draw and label the graphs of the following equations:

   (a) $y = \frac{1}{4}x$ (f) $y = -\frac{1}{3}x$

   (b) $y = \frac{1}{2}x$ (g) $y = -\frac{1}{2}x$

   (c) $y = \frac{2}{3}x$ (h) $y = -\frac{3}{4}x$

   (d) $y = x$ (i) $y = -x$

   (e) $y = 2x$ (j) $y = -2x$

2. Verify your drawing by comparing it with the page of graphs on page 12 at the end of this section. Then answer the following questions about the lines and their equations.

   (a) What point is on all of the lines?

   (b) Which equations have graphs that rise from left to right?

   (c) For the equations having graphs that rise from left to right, is the coefficient of x (the number by which x is multiplied) a positive or a negative number?

   (d) For the equations in which the coefficient of x is positive, what relation do you observe between the value of the coefficient and the steepness of the line?

(e) What facts do you observe about the graphs of equations in which the coefficient of x is negative?

(f) Note that each equation in Exercise 1 is in the form $y = mx$ where m is a real number. Write the equation of the x-axis in this form.

(g) What is the equation of the y-axis? Can it be written in the form $y = mx$?

From the above exercises, the following observations can be made:

(1) If an equation can be written in the form $y = mx$, where m is any real number, its graph is a line in the coordinate plane, through the origin.

(2) For every line through the origin, except the y-axis, there is an equation of the form $y = mx$.

(3) If a line has an equation of the form $y = mx$, the sign of m determines whether the line rises or falls. For $m > 0$, the line rises from left to right; for $m < 0$, the line falls from left to right; for $m = 0$, the line is the x-axis.

(4) The value of $|m|$ determines the steepness of the line; of two equations with different values of m, the one with the larger value of $|m|$ rises or falls more steeply.

In the equation $y = mx$, the real number m is called the slope of the associated line. It determines the direction of the line.

Now let us consider functions whose graphs do not all contain the origin. For example, in each of the functions

(a) $x \to 2x$

(b) $x \to 2x + 4$

(c) $x \to 2x - 5$,

the coefficient of x is 2.

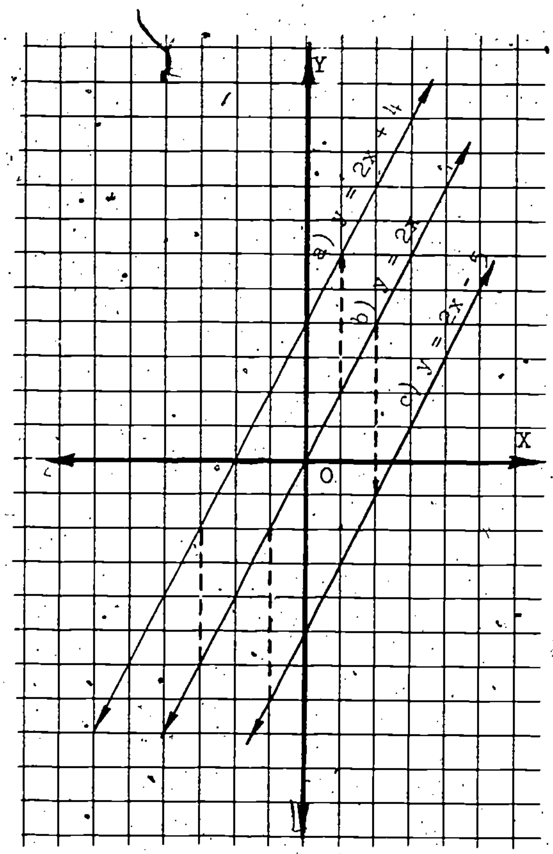
From the graphs of the functions it is apparent that, for any value of x

(1) the value of y in (b) is 4 more than the value of y in (a), and

(2) the value of y in (c) is 5 less than the value of y in (a).

That is, (1) the graph of "$y = 2x + 4$" is the graph of "$y = 2x$" moved <u>upward</u> 4 units, and (2) the graph of "$y = 2x - 5$" is the graph of "$y = 2x$" moved <u>downward</u> 5 units.

We say that all three of the lines have the same <u>slope</u>, 2.



<u>Exercises 17-1b</u>

(Class Discussion)

1. (a) With reference to one set of axes, draw the graphs of these three equations:

$$y = -\frac{2}{3}x$$

$$y = -\frac{2}{3}x + 4$$

$$y = -\frac{2}{3}x - 3.$$

(b) Do the lines have the same slope? What is their slope? Do they appear to be parallel?

2. (a) With reference to one set of axes, draw graphs of:

$$y = 0$$

$$y = -2$$

$$y = 3.$$

(b) Do these lines have the same slope? What is their slope? Do they appear to be parallel?

The conclusions that we have made so far include the following:

(1) The slope of a line is the value of m in the equation of the line, written in the form $y = mx + b$. That is, it is the number which is the coefficient of x when the equation is written in that form.

(2) If the value of m is 0, the equation takes the form "$y = 0 \cdot x + b$" which is usually written "$y = b$". The graphs of such equations are horizontal lines and have zero slope. The slope of any horizontal line is 0.

(3) Equations of the form $x = a$ cannot be put into the form $y = mx + b$. Their graphs are vertical lines. No slope is assigned to a vertical line. Sometimes we express this part by saying that the slope for a vertical line is undefined.

Exercises 17-1c

1. State the slope of the graph of each of the following equations:

(a) $y = x + 207$

(b) $y = -x + \frac{7}{2}$

(c) $3y = 3x + 11$

(d) $3y = 27$

(e) $y = -3x + 1$

(f) $x = -5$

(g) $3y = 7x + 2$

(h) $4y = -2x + 5$

2. From the equations given here, state all those pairs whose graphs have equal slopes and for each pair tell what the slope is.

(a) $3y = 2x + 1$

(b) $y = 9$

(c) $x + 5 = 5$

(d) $y = \frac{2}{3} x$

(e) $y = \frac{3}{2} x + 4$

(f) $y + 1 = 4$

(g) $y = -5 - \frac{2}{3} x$

(h) $2y = 3x + 4$

(i) $x - 2 = 3$

(j) $2x + 3y = 7$

3. Formulas which occur in science, in business, or in everyday life, frequently describe linear functions. For example, the formula $i = .05p$, used in computing simple interest for one year at 5 % on a given principal, is a linear equation. The slope of its graph is .05, or $\frac{5}{100}$. Among the ways in which we might interpret this number are the following:

(1) If the principal is increased by \$1, the interest is increased by 5 cents; or

(2) the interest increases by \$5 for each increase of \$100 in the principal; or

(3) each increase of 20 cents in the principal has a corresponding increase of 1 cent in the interest.

For each of the following formulas, state the slope of the graph of the function and interpret the meaning of the slope in terms of the situation described.

(a) $D = 50t$, a formula for the distance in miles a car travels at 50 miles per hour in a given number of hours.

(b) $F = \frac{9}{5} C + 32$, the formula used in changing a temperature measure from the Centigrade scale to the Fahrenheit scale.

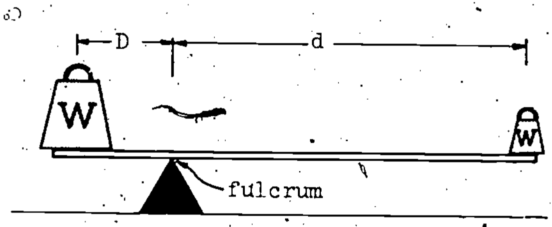
(c) $C = \frac{5}{9} (F - 32)$, the formula for changing temperature measure from degrees Fahrenheit to degrees Centigrade.

(d) $C = .75 + .10 (n - 10)$, $n \geq 10$, a formula for the cost in dollars of a telegram of 10 or more words, if the charge is 75¢ for 10 words and 10¢ for each additional word.

4. The formula for the simple lever is

$$W \times D = w \times d$$

which represents the fact that two weights on opposite sides of the point of suspension (called the fulcrum) balance each other if the product of one weight and its distance from the fulcrum equals the product of the other weight and its distance.



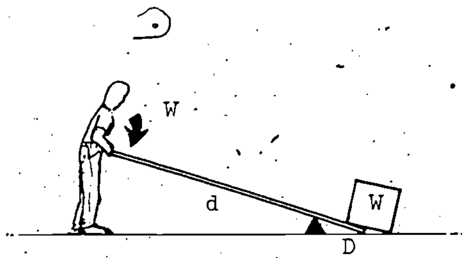
If any two of the four quantities W, D, w, and d have constant values, then an equation can be written which describes a functional relationship between the two remaining variables.

For each case below, write an equation which describes a function, and state whether or not the function described is linear. If it is linear, state the slope of its graph.

(a) A 50-lb. weight is 5 feet from the fulcrum.

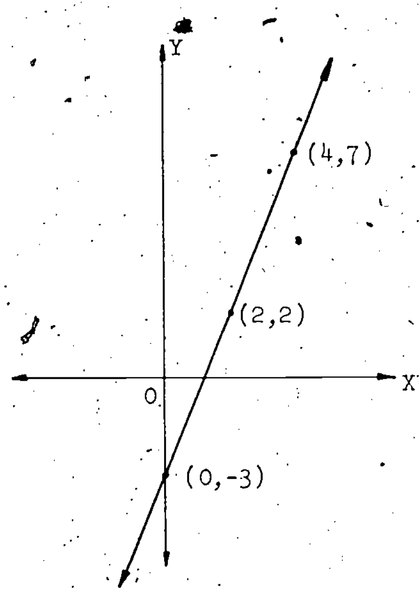
(b) A 50-lb. weight on one side of the fulcrum balances a 60-lb. weight on the opposite side.

(c) The lever can be used to aid in moving heavy objects by pushing down on one end of a bar with the other end under the weight to be lifted, as shown in the sketch.



Assume that the bar is 10 feet long and the fulcrum is placed 2 feet from the weight to be lifted.

If we know two points on a line can we compute its slope? Since for every two distinct points there is exactly one line, and for every non-vertical line there is a real number which is the slope of the line, we should be able to find such a number.



We shall begin by looking at the graph of a particular function,

$$f : x \to \frac{5}{2}x - 3.$$

The points labelled in the figure are on the graph, since each of the ordered pairs $(0,-3)$, $(2,2)$, and $(4,7)$ satisfies the equation. By inspecting the equation $y = \frac{5}{2}x - 3$ we see that the slope of the line is $\frac{5}{2}$. How could we compute the slope if we had known only that the pairs (2,2) and (4,7) named points on the line? We might note that moving from (2,2) to (4,7) could be accomplished by moving from (2,2) to (4,2), and then from (4,2) to (4,7). That is, we could move 2 units <u>to the right</u> and 5 units <u>upward</u>. Since each change is in a <u>positive</u> direction, we can obtain $\frac{5}{2}$, the number which is the <u>slope</u> of the line, from the fraction

$$\frac{\text{difference of ordinates}}{\text{difference of abscissas}}, \text{ or } \frac{7-2}{4-2} = \frac{5}{2}.$$

As a further check, suppose that we had used the pair of points (0,-3) and (2,2). In that case, the fraction would have been

$$\frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \frac{2-(-3)}{2-0} = \frac{5}{2}.$$

If we use the pair of points (0,-3) and (4,7), the fraction is:

$$\frac{7-(-3)}{4-0} = \frac{10}{4} = \frac{5}{2}.$$

Exercises 17-1d

(Class Discussion)

1. Consider the function $f : x \rightarrow -3x + 4$.

   (a) What is the slope of its graph?

   (b) Check that the ordered pairs $A(-1,7)$, $B(2,-2)$, and $C(3,-5)$ satisfy the equation $y = -3x + 4$, hence A, B, and C are points on its graph.

   (c) Locate points A, B, and C on a coordinate plane and draw the graph of the equation.

   (d) Compute the value of the fraction

   $$\frac{\text{difference of ordinates}}{\text{difference of abscissas}}$$

   for each of the following pairs of points.

   For A and B, $\frac{-2-7}{2-(-1)} = \frac{-9}{3} =$ ________

   For A and C, $\frac{-5-7}{3-(-1)} =$ ________

   For B and C, ________

2. What happens if you use the pairs of coordinates in the opposite order? For example, try computing:

For A and B, $\frac{7 - (-2)}{-1 - 2} =$ ______

For A and C, $\frac{7 - (-5)}{-1 - 3} =$ ______

For B and C, $\frac{-2 - (-5)}{2 - 3} =$ ______

All of this leads to the general statement:

If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two points on a non-vertical line, then the slope m of $\overleftrightarrow{P_1 P_2}$ is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

We can convince ourselves ~~ves of~~ the truth of this statement by the following argument:

Since P_1 and P_2 are on the line of which the general equation is

$$y = mx + b,$$

by substituting the coordinates in the equation we get the two true statements

$$y_2 = mx_2 + b$$

$$y_1 = mx_1 + b.$$

Subtracting:

$$y_2 - y_1 = (mx_2 - mx_1) + (b - b).$$

$$y_2 - y_1 = m(x_2 - x_1) + 0.$$

Since the line is not vertical, $x_2 \neq x_1$, hence $x_2 - x_1 \neq 0$, and we can divide both sides by the number $x_2 - x_1$, getting

$$\frac{y_2 - y_1}{x_2 - x_1} = m.$$

Exercises 17-1e

For each pair of points A, B find:

(a) the slope of $\overleftrightarrow{AB}$;

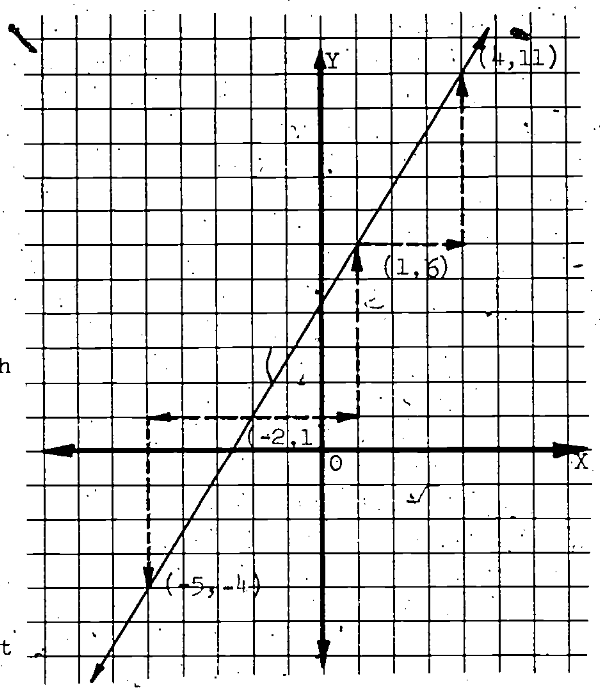
(b) the length of $\overline{AB}$;

(c) the midpoint of $\overline{AB}$.

1. $A(-7,-3)$; $B(5,2)$
2. $A(-7,3)$; $B(8,3)$
3. $A(3,-12)$; $B(-8,10)$
4. $A(-6,5)$; $B(6,0)$
5. $A(0,0)$; $B(-6,-2)$
6. $A(-7,4)$; $B(0,0)$

Now that we know the relation between the slope of a line and the coordinates of two points on the line, we can use this knowledge in various ways. For example, if we know one point on a line and the slope of the line, we can locate other points on the line and thus draw the line very quickly.

Example 1. Draw the line with slope $\frac{5}{3}$ and passing through the point $(-2,1)$.



For any other point (x,y) on the line, we can write

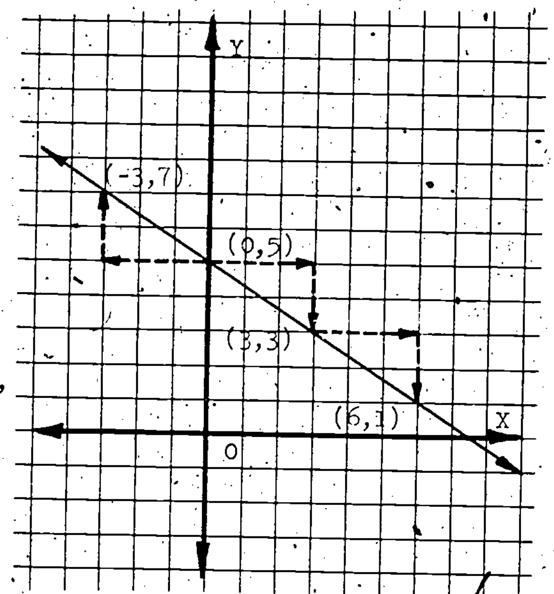
$$\frac{y-1}{x-(-2)} = \frac{5}{3}.$$

If $y - 1 = 5$, $y = 6$.

If $x + 2 = 3$, $x = 1$.

Hence $(1,6)$ is also on the line. In fact, from any point on the graph we can find another point by adding 3 to the abscissa and 5 to the ordinate of the known point. Since $\frac{5}{3} = \frac{-5}{-3}$, we can also find points by adding -3 to the abscissa and -5 to the ordinate. Thus for a line with slope $\frac{5}{3}$, if we know one point we can find as many other points as we choose by repeating the count

"5 to the right and up 3". More points could be reached by counting "5 to the left and down 3".



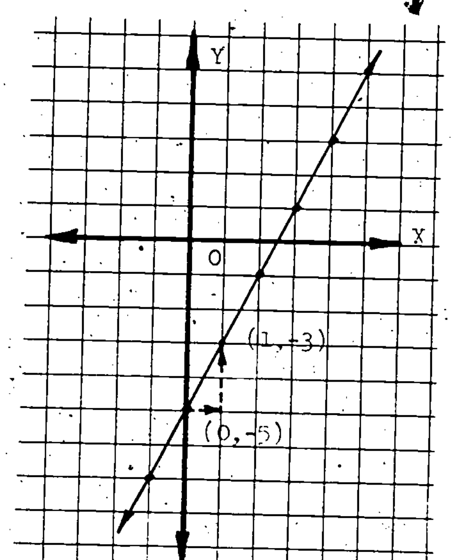
Example 2. Draw the graph of

$$y = -\frac{2}{3}x + 5.$$

Note that the slope is $-\frac{2}{3}$. Since $-\frac{2}{3} = \frac{-2}{3} = \frac{2}{-3}$, this time the count is "3 to the right and down 2" or "3 to the left and up 2". To find a starting point on the graph, we might let $x = 0$, in which case $y = 5$.

The value of y when $x = 0$ is always easy to find. For an equation in the form $y = mx + b$, if $x = 0$, $y = b$. Hence the point $(0,b)$ is frequently used as a starting point. Since it is the point where the graph intersects the Y-axis, we shall call it the y-intersection of the line. The value of y at this point is called the y-intercept of the line.

Thus we see that in an equation of the form $y = mx + b$, the number m is the slope of the line and the number b is its y-intercept. For this reason, $y = mx + b$ is called the "slope-intercept" form of a linear equation.



Example 3. Draw the graph of

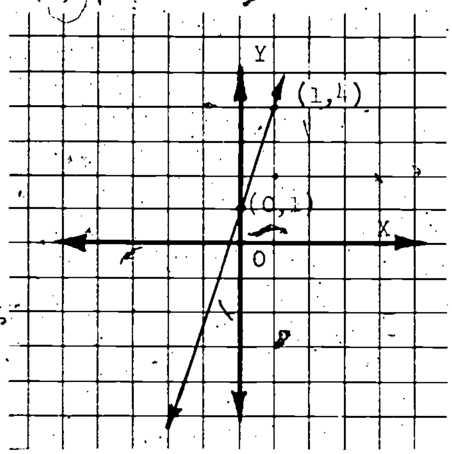
$$y = 2x - 5.$$

Here the y-intercept is -5, so $(0,-5)$ is a convenient starting point. The slope is 2, which might be written in fraction form as $\frac{2}{1}$. From any known point on the line we can find another point by counting "1 to the right and up 2".

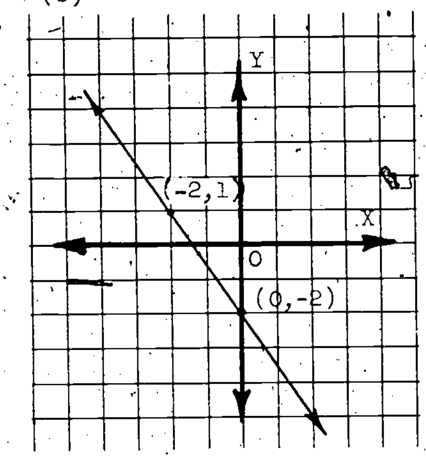
Exercises 17-1f

1. State the slope of each of the following graphs.

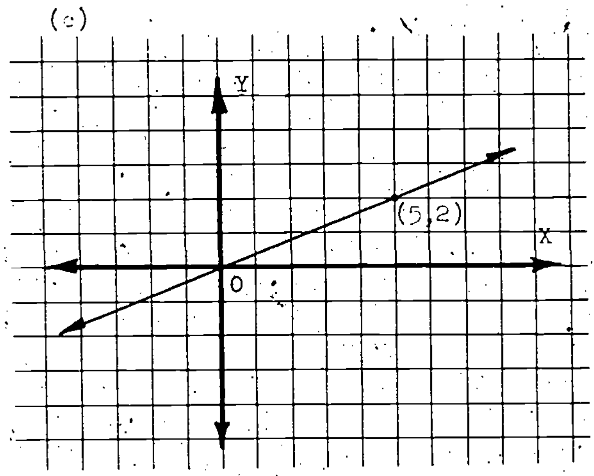
(a)



(b)



(c)



2. For each equation, locate one point on the graph, then use the slope to find other points, and draw the graph of the equation.

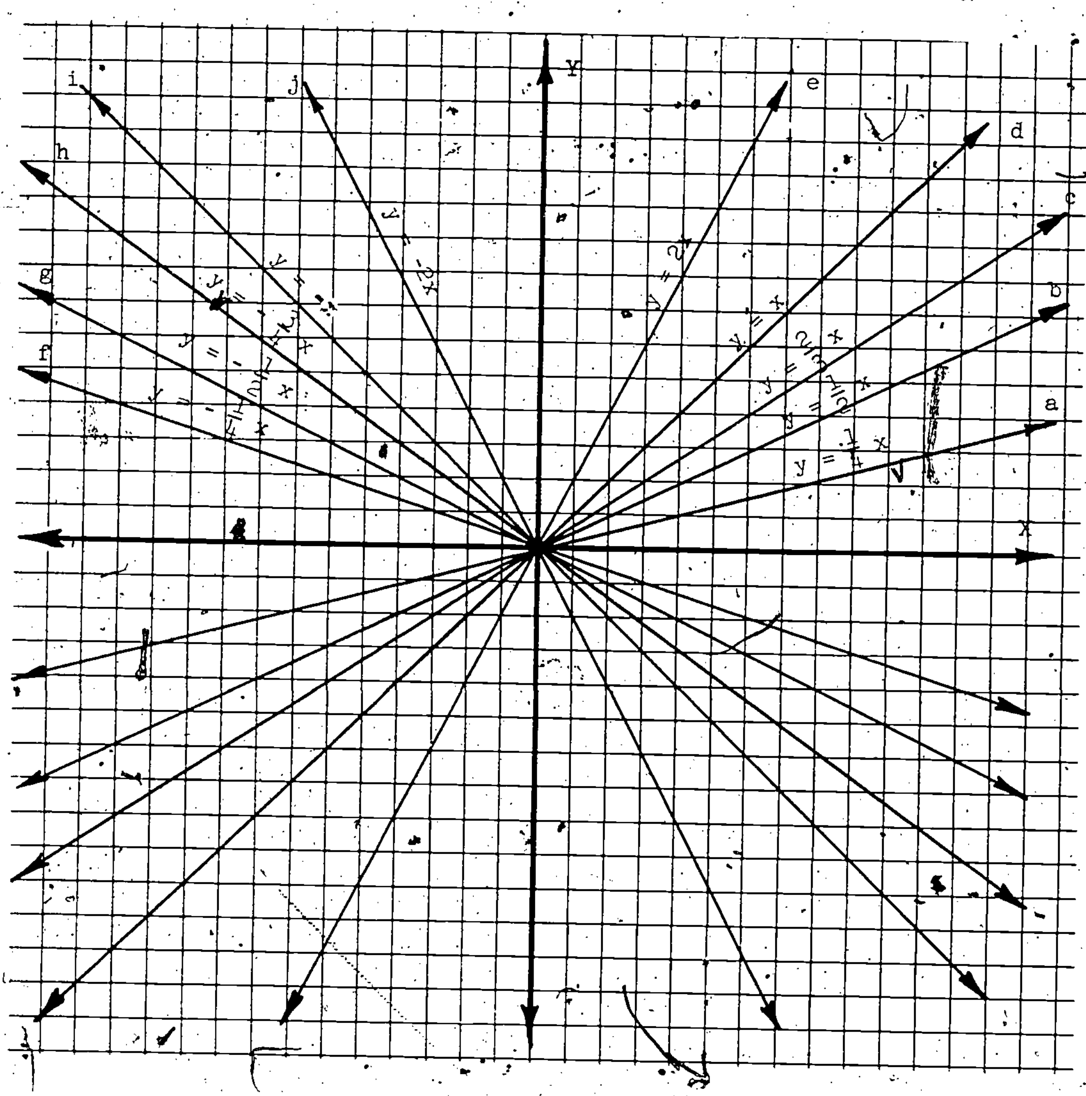
(a) $y = \frac{3}{2}x - 2$

(b) $y = -3x + 7$

(c) $y = \frac{2}{3}x + \frac{1}{3}$

(d) $2x + y = 9$

(e) $3x + 5y - 10 = 0$



Graphs for Exercises 17-1a

17-2. Writing the Equation of a Line

What we have learned about the slopes of lines can also be of use in writing the equation of a line when we know either one point on the line and its slope or two points on the line.

Example 1. Find an equation in the slope-intercept form $y = mx + b$ of the line through $(9,4)$ with slope $\frac{2}{3}$.

Since we know that $m = \frac{2}{3}$, we can write the equation

$$y = \frac{2}{3}x + b.$$

Now, if the point $(9,4)$ is on the line, then its coordinates satisfy the equation of the line, hence

$$4 = \frac{2}{3}(9) + b$$

must be a true sentence. What value of b makes it true?

$$4 = 6 + b$$

is true for $b = -2$.

Thus we now know values for m and b in the equation $y = mx + b$. By substituting these values, we arrive at the desired equation:

$$y = \frac{2}{3}x - 2.$$

Example 2. Find an equation in the form $y = mx + b$ of the line determined by the two points $(-2,5)$ and $(2,-1)$.

Here we begin by calculating the slope from the coordinates of the two known points:

$$m = \frac{-1 - 5}{2 - (-2)} = \frac{-6}{4} = -\frac{3}{2}.$$

Thus $y = -\frac{3}{2}x + b$.

Since one point on the line is $(-2,5)$,

$$5 = -\frac{3}{2}(-2) + b$$

$$5 = 3 + b$$

$$b = 2.$$

Substituting $\frac{-3}{2}$ for m, and 2 for b in $y = mx + b$, we get the equation:

$$y = -\frac{3}{2}x + 2.$$

Verify for yourself the fact that the point $(2,-1)$ is on the graph of $y = -\frac{3}{2}x + 2$.

Exercises 17-2a

1. For each of the following, write an equation in the slope-intercept form, $y = mx + b$.
   (a) The line through $(1,5)$ with slope $\frac{1}{2}$.
   (b) The line through $(2,1)$ with slope $-\frac{5}{2}$.
   (c) The line with slope $\frac{5}{3}$ and y-intercept 5.
   (d) The line with slope 0 and y-intercept -2.
   (e) The line containing points $(0,5)$ and $(8,13)$.
   (f) The line containing points $(0,3)$ and $(3,12)$.
   (g) The line through points $(0,3)$ and $(-2,3)$.
   (h) The line determined by points $(-3,2)$ and $(7,5)$.
2. Consider line ℓ containing the points $(2,3)$ and $(9,5)$. Which of the following points are on the line? [Hint: first determine the slope of ℓ; then, for each point, compare the slope of the line through that point and $(2,3)$ with the slope of ℓ.]
   (a) $(30,11)$
   (b) $(7,4)$
   (c) $(22,9)$
   (d) $(23,9)$
   (e) $(19,58)$
   (f) $(58,19)$

In finding an equation in the form $y = mx + b$ of a line through a given point with a given slope, we might reason as follows:

If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two points on a non-vertical line, the slope m of $\overleftrightarrow{P_1P_2}$ is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Since $x_2 \neq x_1$, we can multiply both sides by $x_2 - x_1$ and get

$$y_2 - y_1 = m(x_2 - x_1).$$

Hence an equation of the line through $P(x_1, y_1)$ with slope m is

$$y - y_1 = m(x - x_1).$$

Example. Write an equation in the form $y = mx + b$ for the line through $(9,4)$ with slope $\frac{2}{3}$.

We know that $(9,4)$ is a point on the line, and that $m = \frac{2}{3}$. Hence we can write

$$y - 4 = \frac{2}{3}(x - 9)$$

$$y - 4 = \frac{2}{3}x - 6$$

$$y = \frac{2}{3}x - 2.$$

Exercises 17-2b

1. Use the formula $y_2 - y_1 = m(x_2 - x_1)$ to find an equation for each of the following:

   (a) The line through $(6,2)$ with slope $\frac{2}{3}$.

   (b) The line through $(3,-3)$ with slope $-\frac{1}{3}$.

   (c) The line through points $(-1,-1)$ and $(1,5)$.

   (d) The line through points $(0,2)$ and $(2,-2)$.

2. Construct the graphs of the following functions, assuming the domain to be $\{x \mid -3 \le x \le 4\}$. [Note that the domain is a closed interval, hence each graph is a segment.]

   (a) $f : x \to x - 2$

   (b) $F : x \to -2x$

   (c) $G : x \to -x - 7$

   (d) $g : x \to \frac{2}{3}x + \frac{1}{3}$

   (e) h : each number x is associated with 5 more than one-half the number.

3. Which of the following are linear functions?

   (a) $f : x \to -(x - 2)$

   (b) $f : x \to |x - 2|$

   (c) $f : x \to |x| - 2$

   (d) $f : x \to (-x) - 2$

   (e) $f : x \to x^2 - 2$

   (f) $f : x \to 2$

   (g) $f : x \to \frac{1}{x - 2}$

4. In science, you will learn about Gay-Lussac's Law concerning the relationship between gas pressure and temperature. You will probably do an experiment planned to convince you of the truth of that Law, which can be stated as follows: For a gas held at a constant volume the pressure (P) of the gas varies linearly with the temperature (T). This can be written as a formula:

$$P = k\,T.$$

In performing an experiment on Gay-Lussac's Law, a student found that the pressure of the gas was 7.5 lbs. per sq. in. at 20° C, and was 9.5 lbs. per sq. in. at 100° C.

(a) Express the student's findings as ordered pairs, (T, P).

(b) Graph the function indicated, using temperature units on the horizontal axis and pressure units on the vertical axis.

(c) Write, in the form $P = mT + b$, an equation which describes the function.

(d) What would be the pressure of the gas at 50° C? At 0° C?

(e) At what temperature would the pressure be 14 lbs. per sq. in.? 0 lb. per sq. in.?

5. On the graph of the equation $y = 2x + \frac{2}{3}$ there is <u>no</u> point (x,y) such that x and y are both integers.

(a) Explain why this is true.

(b) Write two other equations for which the same statement is true.

6. Construct a flow chart for finding the slope of a line, given the coordinates of two different points on the line.

17-3. Parallel and Perpendicular Lines

Earlier in this chapter we mentioned that it is possible to prove that the graph of any equation of the form $y = mx + b$ is a straight line. In deciding how to proceed with the proof, we shall first consider a particular equation of that form and the graph of the equation.

Exercises 17-3a

(Class Discussion)

1. For the equation $y = 2x + 3$,

   (a) When, $x = 0$, $y = \underline{\ ?\ }$

   (b) When $x = 1$, $y = \underline{\ ?\ }$

2. Since the ordered pairs $(0,3)$ and $(1,5)$ satisfy the equation $y = 2x + 3$, its graph includes the points $P(0,3)$ and $Q(1,5)$. Locate the points and draw $\overleftrightarrow{PQ}$; construct $\overleftrightarrow{PR}$ parallel to the x-axis, and $\overleftrightarrow{RQ} \perp \overleftrightarrow{PR}$. Check your figure with that shown here. Verify that

   $$\frac{RQ}{PR} = \frac{2}{1} = 2.$$

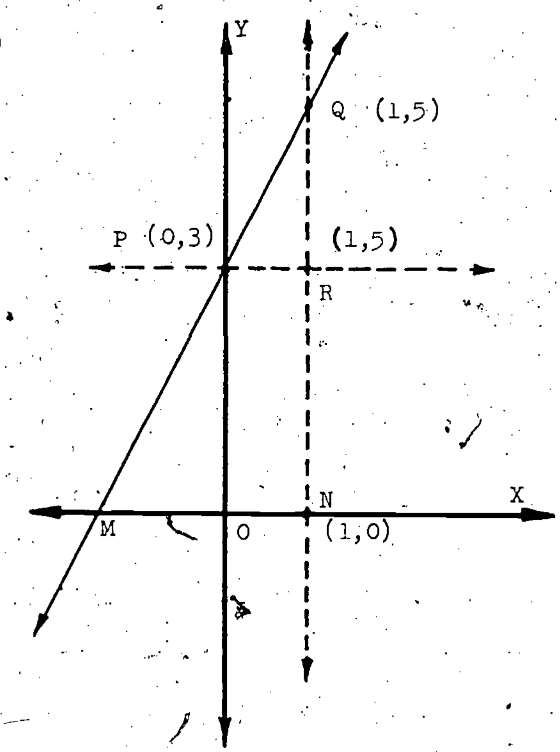


Figure 1

3. Select some other point $S(x,y)$ on $\overleftrightarrow{PQ}$. Draw $\overleftrightarrow{ST} \perp \overleftrightarrow{PR}$ at $T(x,3)$. Represent the length of $\overline{ST}$ by z. Check your figure now with that shown here.

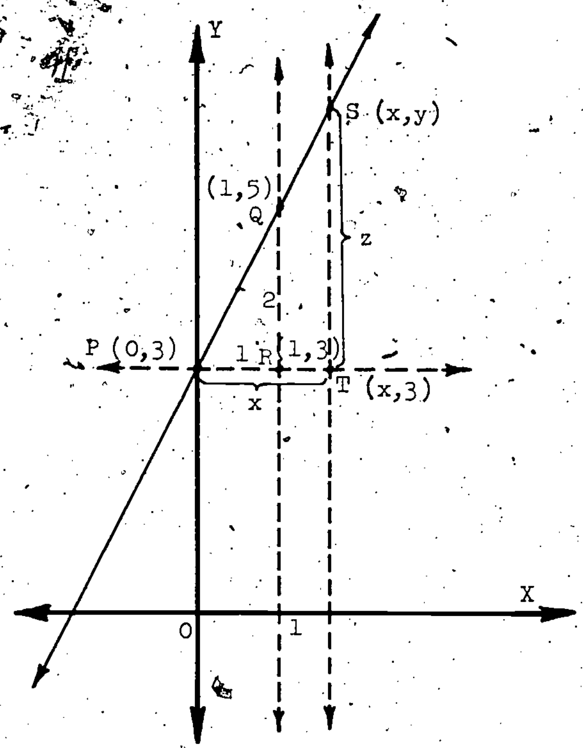


Figure 2

4. Use the geometry of similar triangles to prove $\triangle QPR \sim \triangle SPT$, thus showing that

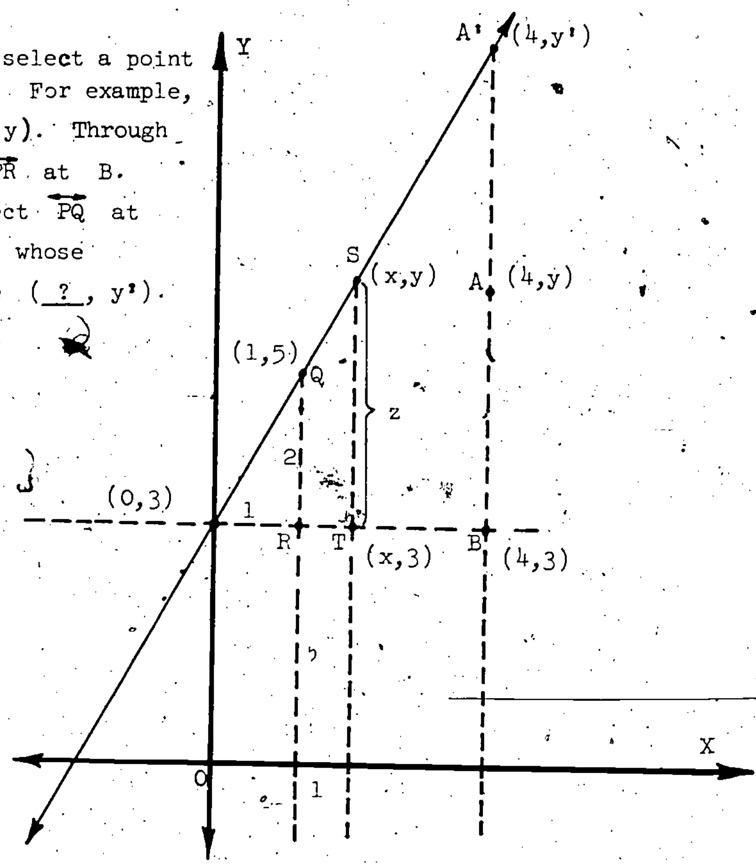
$$\frac{z}{x} = \frac{2}{1}$$

$$z = \underline{\quad ? \quad}$$

$$y = z + 3 = \underline{\quad ? \quad}.$$

5. S represents any point on $\overleftrightarrow{PQ}$ other than P or Q, and the coordinates (x,y) of S satisfy the equation $y = 2x + 3$. So far we have shown that if a point is on $\overleftrightarrow{PQ}$, then its coordinates satisfy the equation $y = 2x + 3$. To complete the proof that the line is the graph of the equation, we must also show that every point (x,y) for which the sentence "$y = 2x + 3$" is true must be on the line. To do this we select a point not on the line and show that if its coordinates satisfy the equation there is a contradiction.

6. On your figure select a point A not on $\overleftrightarrow{PQ}$. For example, let A be $(4,y)$. Through A draw $\overleftrightarrow{AB} \perp \overleftrightarrow{PR}$ at B. It will intersect $\overleftrightarrow{PQ}$ at some point A' whose coordinates are $(\underline{\ ?\ }, y')$.



7. Assume that the coordinates of A satisfy the equation $y = 2x + 3$. Then

$$y = \underline{\ \ ?\ \ }, \text{ and A is the point } (4, \underline{\ ?\ }).$$

8. However, $A'(4,y')$ is on $\overleftrightarrow{PQ}$, and we have proved that if a point is on $\overleftrightarrow{PQ}$, then its coordinates satisfy the equation $y = 2x + 3$. Hence

$$y' = 2(4) + 3 = 11,$$

and

$$A' \text{ is the point } (\underline{\ ?\ }, \underline{\ ?\ }).$$

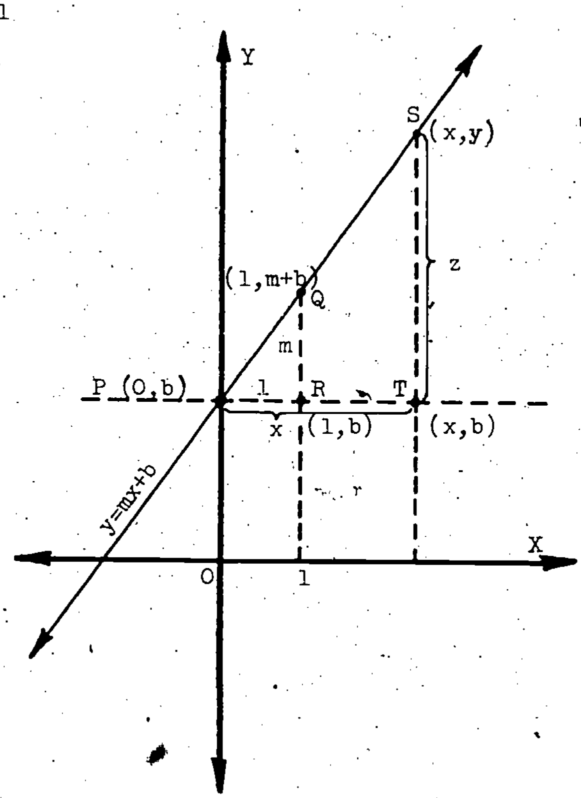
We have shown that the coordinates of A', a point on $\overleftrightarrow{PQ}$, are the same as those of A. Hence A and A' are the same point. This contradicts our assumption that A is a point not on $\overleftrightarrow{PQ}$. Thus if the coordinates of a point satisfy the equation $y = \underline{\qquad}$, then the point is on the line which is determined by two ordered pairs that satisfy the equation.

Through the sequence of exercises above, we have shown that the graph of $y = 2x + 3$ is a straight line.

We could use exactly the same sort of argument for each individual equation of the form $y = mx + b$, to show that the graph of that particular equation was a straight line. It would save time, however, to go through the proof once, for the general equation

$$y = mx + b, \quad m \neq 0$$

using the generalized figure shown here.



The proof would begin like this: $P(0,b)$ and $Q(1,m+b)$ are two points on the graph of $y = mx + b$, since both pairs of coordinates satisfy the equation.

If $S(x,y)$ is a point on $\overleftrightarrow{PQ}$, then we can prove that $\triangle STP \sim \triangle QRP$, and hence that

$$\frac{z}{x} = \frac{m}{1}$$

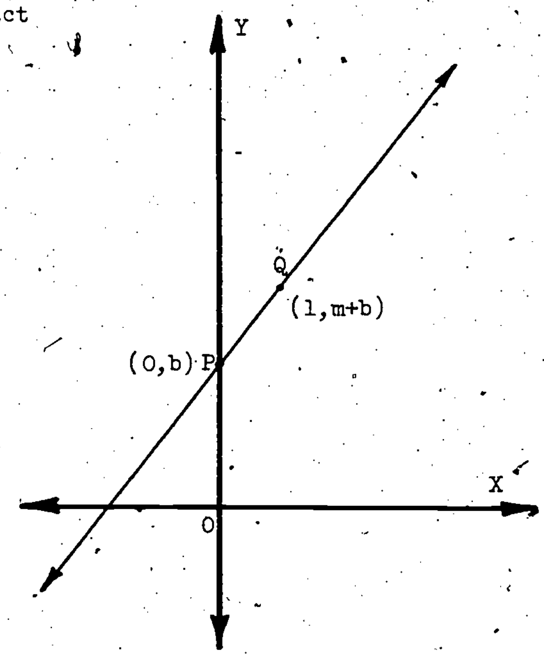
$$z = mx$$

$$y = mx + b.$$

Thus the coordinates (x,y) of any points on $\overleftrightarrow{PQ}$ satisfy the equation $y = mx + b$.

Exercises 17-3b

1. Complete the proof of the fact that the graph of $y = mx + b$, $m \neq 0$, is a straight line by proving that if point A is <u>not</u> on $\overleftrightarrow{PQ}$, the assumption that its coordinates satisfy the equation leads to a contradiction (see Exercises 17-3a, 6-8).



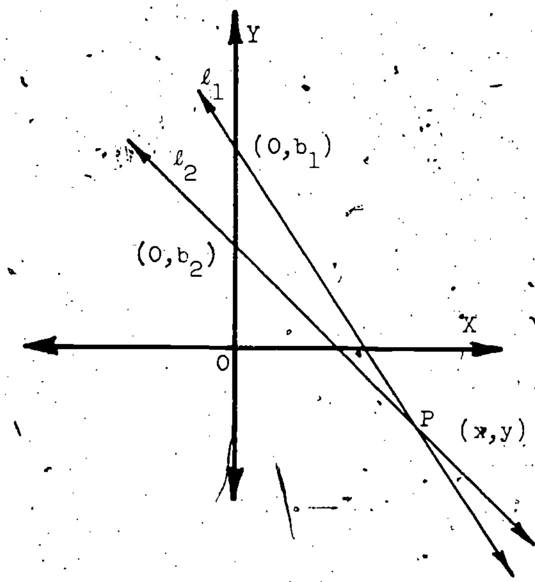
2. For the equation $y = mx + b$, if $m = 0$, then $y = 0 \cdot x + b = b$. In the coordinate plane, is the graph of $y = b$ a line? Explain.

3. Is the graph, in the coordinate plane, of the equation $x = k$ a line? Explain.

We shall state the fact that has been proved as the first of our fundamental theorems in this chapter:

Theorem 17-1. The graph of any equation of the form $y = mx + b$ is a straight line.

You have probably noticed that two distinct nonvertical lines with the same slope seem to be parallel. In fact, we can prove very simply that this is true. Remember that if we have two lines in the same plane, the lines are either parallel or they intersect. We shall prove that if the slopes are equal then the lines cannot intersect.



Suppose that ℓ_1 and ℓ_2 are two distinct lines whose equations are

$$\ell_1 : y = mx + b_1$$

$$\ell_2 : y = mx + b_2,$$

where $b_1 \neq b_2$.

If the lines intersect at some point $P(x,y)$, then at that point the ordinates (values of y) are equal, so

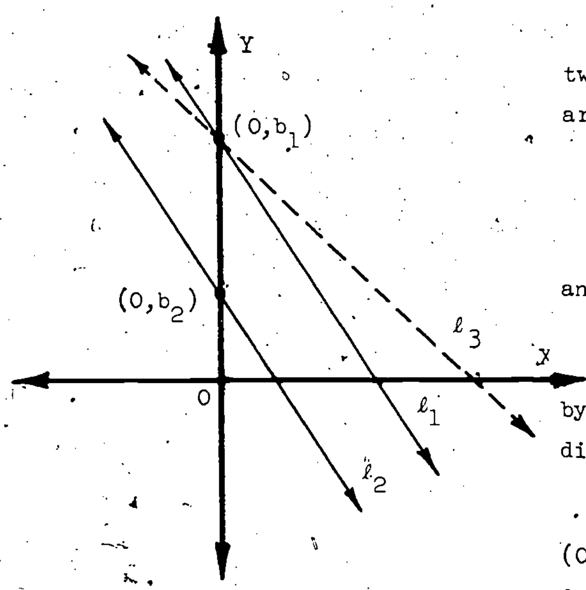
$$mx + b_1 = mx + b_2.$$

Hence $$b_1 = b_2,$$

which contradicts the assumption that $b_1 \neq b_2$.

We have proved that for two distinct lines with the same slope to intersect leads to a contradiction. If they cannot intersect, they must be parallel.

We can use a similar argument to prove the converse of this, that is, if two nonvertical lines are parallel then their slopes are equal.



Suppose that ℓ_1 and ℓ_2 are two distinct lines whose equations are

$$\ell_1 : y = m_1x + b_1$$

$$\ell_2 : y = m_2x + b_2,$$

and that $\ell_1 \parallel \ell_2$.

We shall prove that $m_1 = m_2$ by showing that we have a contradiction if $m_1 \neq m_2$.

The y-intersection of ℓ_1 is $(0,b_1)$. Through that point we can draw a line ℓ_3 whose equation is

$$y = m_2x + b_1.$$

Now, $\ell_3 \parallel \ell_2$, because we have proved that if two distinct lines have the same slope, then the lines are parallel.

Thus we have two lines, ℓ_1 and ℓ_3, through the point $(0, b_1)$, both parallel to ℓ_2. This contradicts the Parallel Postulate, which states that through a given point not on a line there can be only one line parallel to the given line.

Since $m_1 \neq m_2$ leads to this contradiction, it follows that it cannot be true, so $m_1 = m_2$ must be true.

Taken together, the two facts just proved constitute a theorem which can be stated as follows:

Theorem 17-2. Two nonvertical lines are parallel if and only if they have the same slope.

Check Your Reading

1. What did we prove about two lines with the same slope?
2. If two lines are in the same plane and are not parallel, what must be true about them?
3. The method that we used is sometimes called "Proof by Contradiction". In this case, what fact is contradicted if the lines are assumed to intersect?
4. What did we prove about two nonvertical parallel lines?
5. How did we use Proof by Contradiction to prove that two nonvertical parallel lines have the same slope?
6. Stated as a single theorem, what has been proved about two nonvertical parallel lines?

Suppose that two lines are vertical? Such lines have no slope, but they are parallel to the y-axis. Thus any two vertical lines are parallel.

Exercises 17-3c

1. For each of the following pairs of equations, find a value of p so that the graph of the first equation will be a line parallel to the graph of the second equation.

   (a) $y = px + 3; \quad y = x$

   (b) $y = px - 2; \quad y = -2x$

   (c) $y = px + \frac{7}{2}; \quad 2y = 7x - 10$

   (d) $y = px + 9; \quad y = 3$

   (e) $y = px; \quad 3x + y = 4$

2. Which of the following are the equations of lines parallel to the graph of $2x - 3y = 1$?

   (a) $y = \frac{2}{3}x$

   (b) $y = -\frac{3}{2}x + 2$

   (c) $3x - 2y = 1$

   (d) $x - \frac{3}{2}y = 4$

   (e) $\frac{2}{3}x = y - 5$

3. The vertices of $\triangle ABC$ are: $A(-2,3)$, $B(5,7)$, $C(3,-1)$. For each statement below, find a value of k that makes the statement true.

   (a) The graph of $y = kx + 7$ is parallel to $\overline{AB}$.

   (b) The graph of $y = kx$ is parallel to $\overline{AC}$.

   (c) The graph of $kx + y + 5 = 0$ is parallel to $\overline{BC}$.

4. The vertices of the quadrilateral $ABCD$ are $A(2,1)$, $B(3,5)$, $C(-5,1)$, $D(-6,-3)$. Prove that $ABCD$ is a parallelogram.

We have seen that the apparent relationship between the slopes of lines and their parallelism is, in fact, true. Now let us look at some other lines and their slopes.

Exercises 17-3d

(Class Discussion)

1. On one set of coordinate axes, draw the graphs of the following equations.

   (a) $y = -3x + 12$

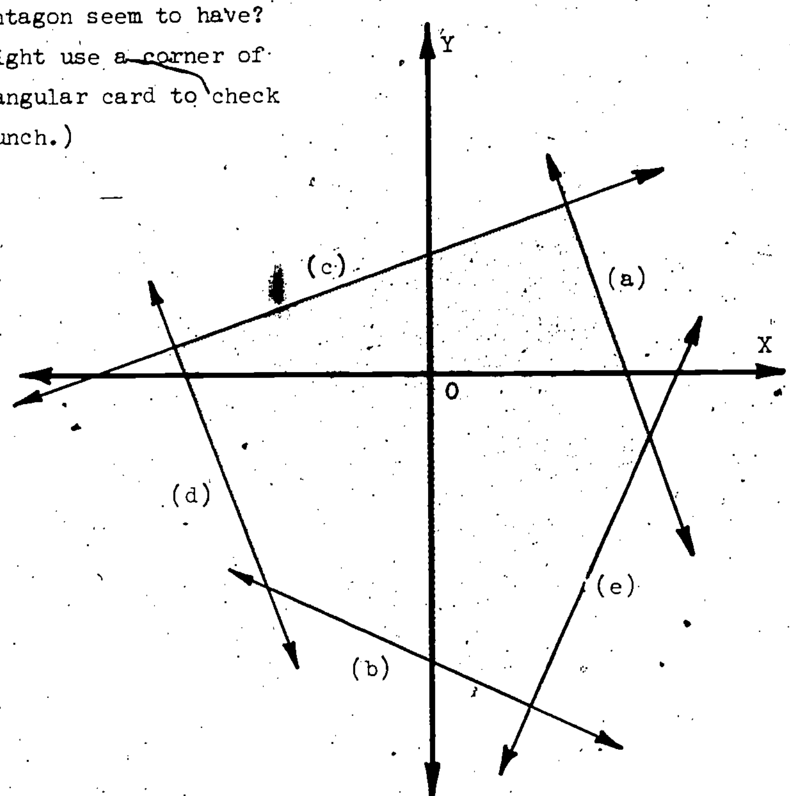
   (b) $y = -\frac{1}{2}x - 6$

   (c) $y = \frac{1}{3}x + \frac{7}{3}$

   (d) $y = -3x - 15$

   (e) $y = 2x - 11$

2. The region bounded by segments of the above lines is shaped like a pentagon, as shown by the shaded area in this figure. How many right angles does the pentagon seem to have? (You might use a corner of a rectangular card to check your hunch.)



3. Refer to Exercises 1 and 2 above to complete the following table concerning pairs of lines which appear to be perpendicular, the equations of the lines, and their slopes.

	Lines	Equations	Slopes	Product of Slopes
(1)	(a)	______	______	__________
	(c)	______	______	
(2)	(d)	______	______	__________
	(c)	______	______	
(3)	(b)	______	______	__________
	(e)	______	______	

4. Complete the following statement based upon the table above:

Two nonvertical lines are __________ if and only if the product of their slopes is __________.

The statement in Exercise 4 above was based upon a small number of cases. However, it is a rather well-known theorem. Let us have a look at a proof of it. Remember that the phrase "if and only if" indicates that we must prove two things:

(1) If the lines are perpendicular, then the product of their slopes is -1, and

(2) if the product of the slopes is -1, then the lines are perpendicular.

Proof of (1):

We look first at two lines ℓ_1 and ℓ_2 through the origin with equations:

$$\ell_1 : y = m_1 x$$

$$\ell_2 : y = m_2 x$$

and such that $\ell_1 \perp \ell_2$.

At the point $R(1,0)$ we construct $\overline{RP} \perp$ to the X-axis, intersecting ℓ_1 at $P(1,m_1)$.

We also construct $\triangle$ OSQ by making $\overline{OS}$ a segment of the X-axis such that $OS = m_1$ and then constructing $\overline{SQ} \perp \overline{OS}$, with Q on ℓ_2.

Now we can show that $\angle QOS = \angle POR$, and thus that rt $\triangle$ OSQ $\cong$ rt $\triangle$ PRO.

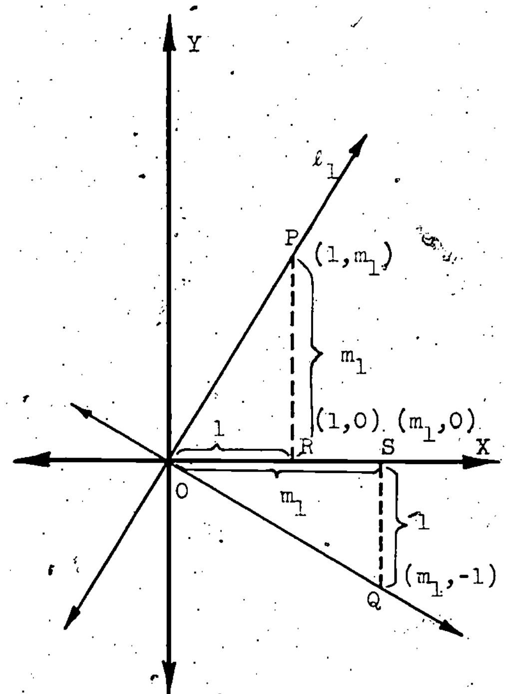


Figure 1

Hence $SQ = OR = 1$, and point Q has coordinates $(m_1,-1)$.

The slope of ℓ_1 is m_1, and the slope of ℓ_2 is $m_2 = \frac{-1-0}{m_1 - 0} = -\frac{1}{m_1}$.

Thus

$$m_2 = -\frac{1}{m_1}$$

and

$$m_1 \cdot m_2 = -1.$$

Suppose that the two lines intersect at a point other than the origin, as in the figure shown here, in which lines

$k_1 : y = m_1 x + b$

$k_2 : y = m_2 x + b$

intersect at T.

Then lines $\ell_1 \parallel k_1$ and $\ell_2 \parallel k_2$ can be constructed through the origin.

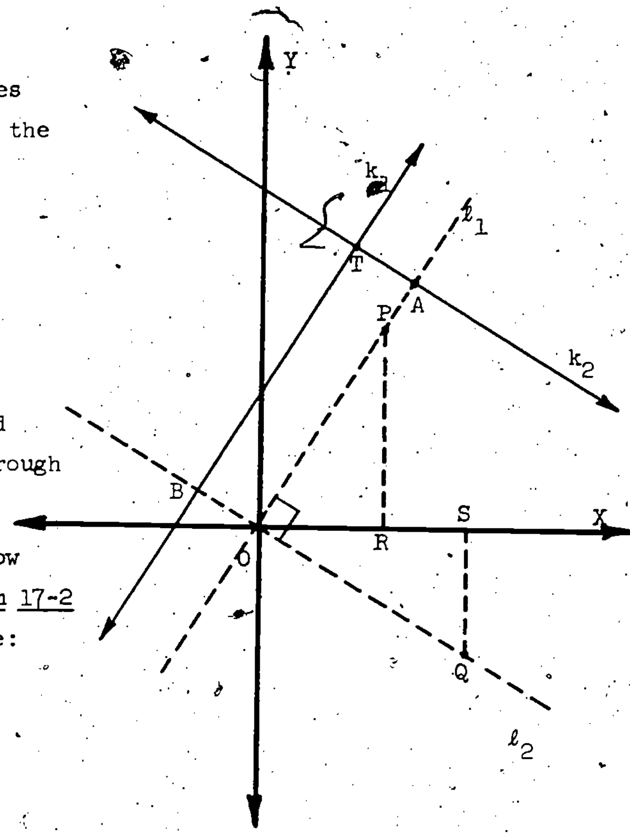
From geometry, we can show that $\ell_1 \perp \ell_2$, and from Theorem 17-2 we know that their equations are:

$\ell_1 : y = m_1 x$

$\ell_2 : y = m_2 x$.

Since ℓ_1 and ℓ_2 are lines through the origin and $\ell_1 \perp \ell_2$, we know that

$m_1 \cdot m_2 = -1$.



Thus we have proved that if any two nonvertical lines are perpendicular, the product of their slopes is -1.

Check Your Reading

1. For lines ℓ_1 and ℓ_2 in Figure 1, what are the equations of the lines? What is the relation of ℓ_1 to ℓ_2?

2. Explain how to prove that $\triangle OSQ \cong \triangle PRO$.

3. For Figure 2, how are ℓ_1 and ℓ_2 constructed? Explain how it follows that, if $k_1 \perp k_2$ then $\ell_1 \perp \ell_2$.

4. If the slope of k_1 is m_1 and if $k_1 \parallel \ell_1$, what is the slope of ℓ_1? How do you know?

5. What have we proved about the slopes of two nonvertical lines that are perpendicular to each other?

The proof of Part (2), that if the product of the slopes of two lines is -1, then the lines are perpendicular, will be asked for as one exercise in Exercises 17-3e.

We state the following theorem about perpendicular lines:

> Theorem 17-3. Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

Exercises 17-3e

1. For each pair of equations, determine a value of m in the first equation such that its graph is perpendicular to the graph of the second equation.

   (a) $y = mx - 4$ and $y = 3x + 2$

   (b) $y = mx$ and $y = -\frac{1}{2}x - \frac{1}{2}$

   (c) $y = mx + 1$ and $2x + y = 3$

   (d) $y = mx - 2$ and $y = \frac{2}{3}x$

2. Which equations given below have graphs perpendicular to the graph of $y = \frac{1}{2}x$?

   (a) $y = 2 - 2x$

   (b) $x + 2y = 3$

   (c) $x - 2y = 3$

   (d) $2y = -4x + 1$

   (e) $y = 2$

3. Write out a proof of the second part of Theorem 17-3. That is, prove that if the product of the slopes of two lines is -1, then the lines are perpendicular. (Hint: first prove this for two lines through the origin, constructing line ℓ_2 in Figure 1 on page 27 in such a way as to make $\triangle OSQ \cong \triangle PRO$.)

4. Since $m_1 \cdot m_2 = -1$ and $m_2 = -\frac{1}{m_1}$ are true for the same values of m_1 and m_2, we could replace "the product of the slopes is -1" by "one slope is the negative of the reciprocal of the other slope". Rewrite Theorem 17-3, making this replacement.

5. Suppose that one of two perpendicular lines is a vertical line.

   (a) What must be true of the other line?

   (b) Why are vertical lines excluded from the statement of the theorem about perpendicular lines and their slopes?

6. Consider $\triangle ABC$ with vertices $A(2,3)$, $B(5,9)$ and $C(11,6)$:

(a) Find the lengths of the three sides.

(b) Find the slopes of the sides.

(c) What kind of triangle is $\triangle ABC$?

(d) Prove that the line joining the midpoints of segments $\overline{AB}$ and $\overline{BC}$ is parallel to $\overline{AC}$.

7. Show that the quadrilateral joining $A(-2,2)$, $B(2,-2)$, $C(4,2)$, and $D(2,4)$ is a trapezoid with perpendicular diagonals. (A trapezoid is a quadrilateral with one pair of parallel sides.)

8. Show that the quadrilateral ABCD with vertices $A(2,1)$, $B(7,1)$, $C(10,5)$, $D(5,5)$ is a rhombus with perpendicular diagonals. (A rhombus is a parallelogram with two adjacent sides congruent.)

9. Given points $A(-7,3)$ and $B(9,-9)$, write the equation of the perpendicular bisector of $\overline{AB}$. (<u>Hint</u>: What is the slope of $\overline{AB}$? What is its midpoint?)

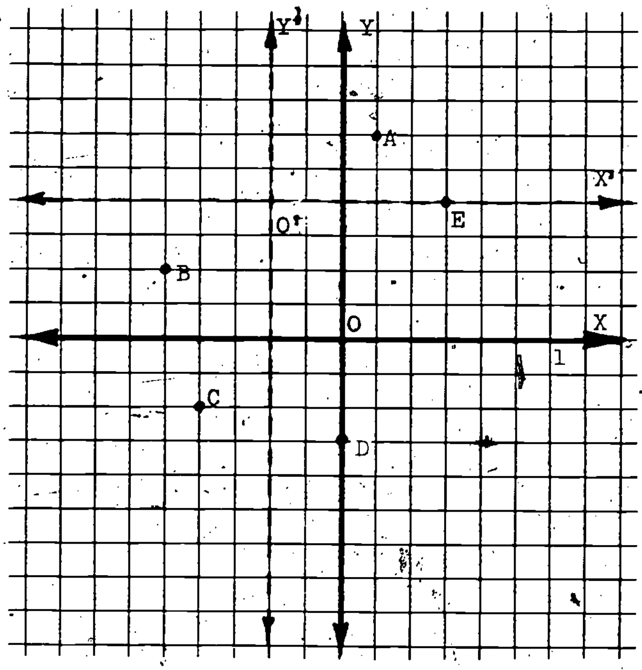
17-4. Translation of Axes; Coordinate Proofs of Geometric Properties

Now suppose that we look at some figures in the number plane and see what happens to them when we shift the coordinate axes. We shall restrict our shifts to ones which keep the axes parallel to themselves. That is, the Y-axis will remain vertical and the X-axis will remain horizontal. Such shifts of axes are called *translations*.

Exercises 17-4a

(Class Discussion)

1. The figure here is made up of five points, A, B, C, D, E. We have indicated two sets of coordinate axes, the (X,Y)-axes and the (X',Y')-axes.



   (a) Give the coordinates of each point with respect to the (X,Y)-axes.

   (b) With respect to the (X',Y')-axes, what are the coordinates of each point?

   (c) For each point, what was the relation of X' to X? of Y' to Y?

   (d) If a point P has coordinates (a,b) with respect to the (X,Y)-axes, what are its coordinates with respect to the (X',Y')-axes?

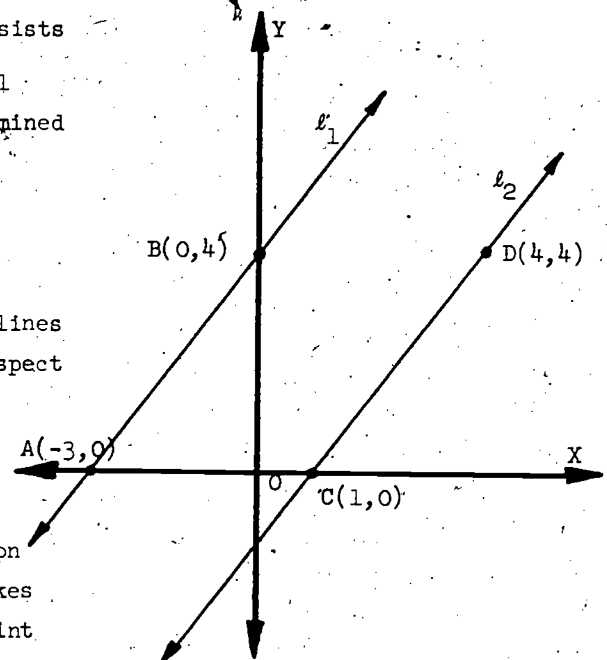
   (e) Give the (X',Y') coordinates of a point whose (X,Y) coordinates are (7,-4).

   (f) Give the (X,Y) coordinates of a point whose (X',Y') coordinates are (7,-4).

   (g) Think of the (X',Y')-axes as representing a slide, or translation, of the (X,Y)-axes, and describe the slide.

(h) What are the (X,Y) coordinates of the new origin?

2. The figure shown here consists of two parallel lines, ℓ_1 and ℓ_2, with ℓ_1 determined by A and B and ℓ_2 determined by C and D, as shown.



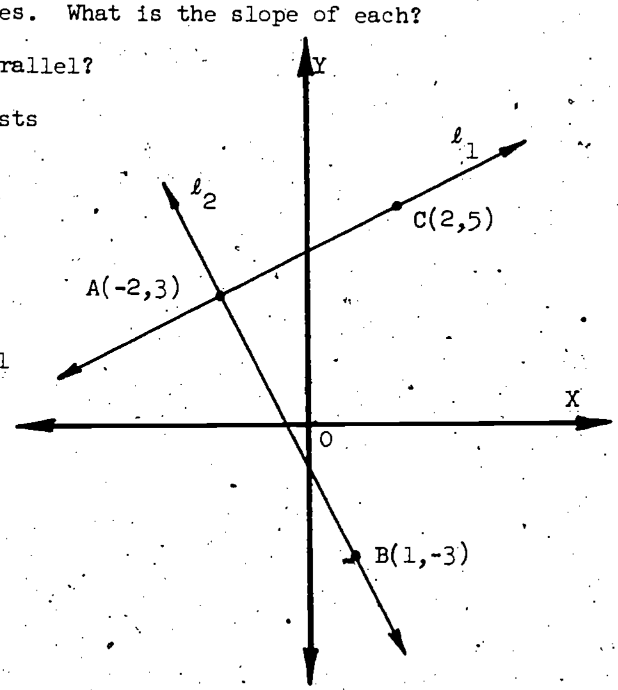
(a) Write equations for lines ℓ_1 and ℓ_2 with respect to the (X,Y)-axes given. What is the slope of each line?

(b) Consider a translation of the axes which takes the origin to the point whose (X,Y) coordinates are $(4,-3)$. What are the new coordinates of points A, B, C, and D?

(c) Write equations for lines ℓ_1 and ℓ_2 with respect to the new position of the axes. What is the slope of each?

(d) Are the lines still parallel?

3. The figure shown here consists of two lines ℓ_1 and ℓ_2, intersecting at $A(-2,3)$. Point $B(1,-3)$ is on ℓ_2 and $C(2,5)$ is on ℓ_1.



(a) Write equations for ℓ_1 and ℓ_2. What is the slope of each?

(b) Prove that $\ell_1 \perp \ell_2$.

(c) Find the lengths of $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$.

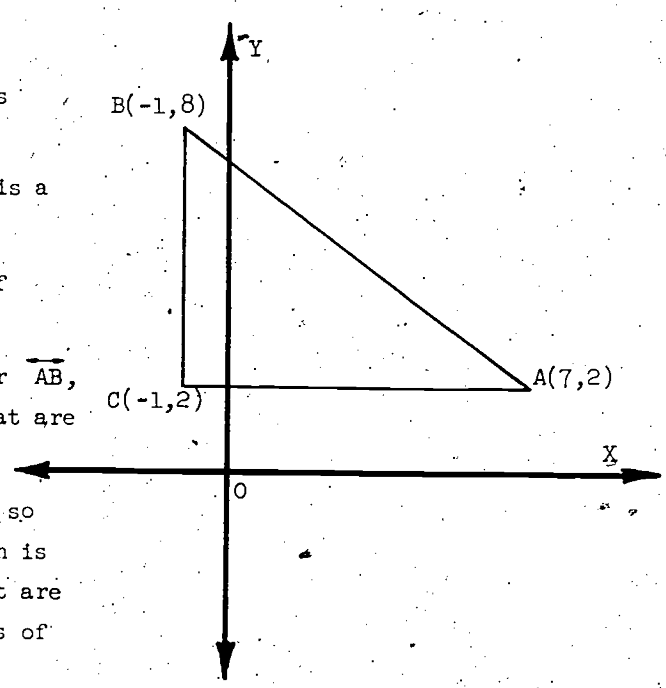
(d) Consider a translation of the axes which moves the X-axis up 2 units and moves the Y-axis one unit to the right. What are the new coordinates of A, B, and C?

(e) What are equations for ℓ_1 and ℓ_2 with respect to the new position of the axes?

(f) Are ℓ_1 and ℓ_2 perpendicular to each other? Justify your answer.

(g) Using the new coordinates of A, B, and C, find the lengths of $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$. Compare these with the lengths found in Part (c).

4. The figure this time is $\triangle ABC$, with vertices as indicated.



(a) Show that $\triangle ABC$ is a right triangle.

(b) Find the lengths of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

(c) Write equations for $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{CA}$. What are their slopes?

(d) Translate the axes so that the new origin is at point C. What are the new coordinates of the vertices?

(e) Using the new coordinates for A, B, and C, what are the lengths of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$?

(f) What are new equations for $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{CA}$?

5. When a property remains unchanged in spite of a transformation, such as a change of axes, we say that the property is <u>invariant</u> under the transformation. In Exercises 1 through 4 above, which of the following properties were invariant under the translations described for the axes?

(a) Coordinates of points

(b) Distances between points

(c) Equations of lines

(d) Slopes of lines

(e) Parallelism of lines

(f) Perpendicularity of lines

The theorems proved in Section 17-3 and the formulas for the length and the midpoint of a segment make it possible for us to prove geometric facts. For convenience, we restate here the formulas and theorems on which we shall base our proofs:

Distance Formula. For points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$,

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Midpoint Formula. For points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, the coordinates of M, the midpoint of $\overline{P_1P_2}$, are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

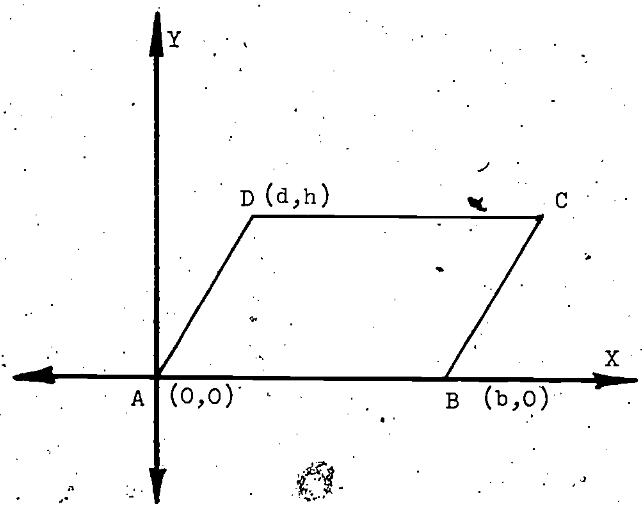
Theorem 17-1. The graph of any equation of the form $y = mx + b$ is a straight line.

Theorem 17-2. Two nonvertical lines are parallel if and only if they have the same slope.

Theorem 17-3. Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

Example 1. Prove that the diagonals of a parallelogram bisect each other.

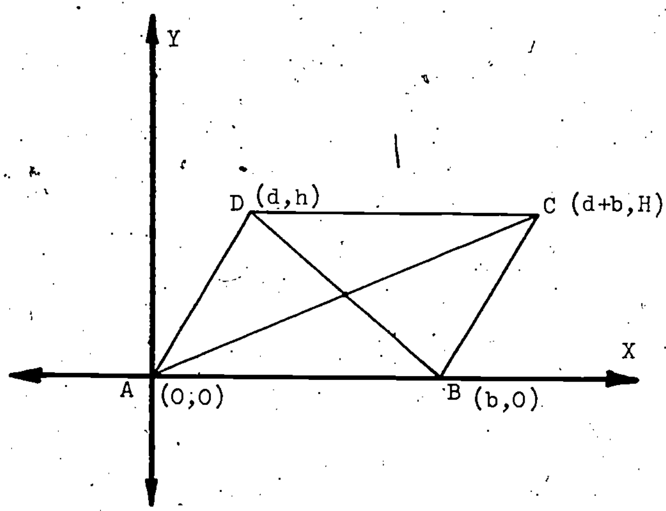
This theorem, which you have already proved by using congruent triangles, can also be proved by assigning coordinate axes to a general parallelogram. Since we can place the axes wherever we choose, we shall place them with the origin at one vertex of the parallelogram, as in the figure shown here. Then the coordinates of A are (0,0), and the second coordinate of B is 0.



We have chosen to use $\underline{b}$ as the first coordinate of B, and to use the letters $\underline{d}$ and $\underline{h}$ as coordinates of D. So B is the point (b,0), and D is the point (d,h). Now we shall assign suitable coordinates to C, making use of the fact that ABCD is a parallelogram.

Since $\overline{DC} \parallel \overline{AB}$, the second coordinate of C is $\underline{h}$, the same as the second coordinate of D. Since the opposite sides of a parallelogram are congruent, DC = AB = b. Hence the first coordinate of C is d + b.

The figure with appropriate coordinates assigned to the vertices, and with the diagonals drawn, is as shown here. We shall prove that $\overline{AC}$ and $\overline{BD}$ bisect each other by showing that their midpoints have the same coordinates.



From the Midpoint Formula, we find:

$$\text{Midpoint of } \overline{AC} = \left(\frac{0 + (d+b)}{2}, \frac{0 + h}{2}\right)$$

$$= \left(\frac{d + b}{2}, \frac{h}{2}\right).$$

$$\text{Midpoint of } \overline{BD} = \left(\frac{d + b}{2}, \frac{h}{2}\right).$$

Thus the two segments have a common midpoint. That is, they bisect each other.

Check Your Reading

1. What geometric theorem was proved in Example 1?

2. In assigning a coordinate system to the figure, where did we place the origin?

3. What coordinates did we assign to points A, B, and D?

4. How did we decide that the coordinates of point C should be $(d + b, h)$?

5. How was the Midpoint Formula used in the proof?

Exercises 17-4b

(Class Discussion)

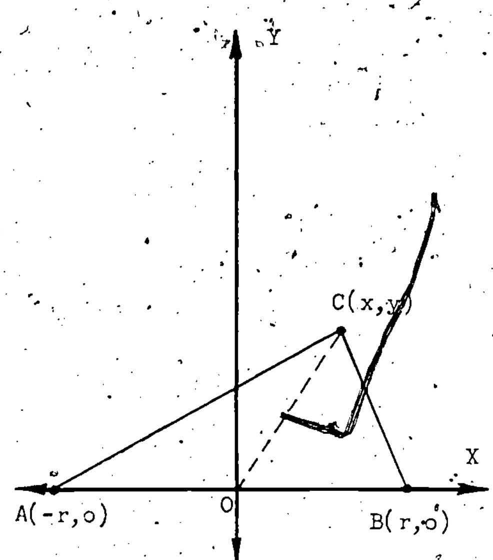
1. As another example of a geometric proof using coordinates, consider the following:

   (a) On a set of coordinate axes draw $\triangle ABC$ with $A(-r,0)$, $B(r,0)$, and C a point (x,y), such that $x^2 + y^2 = r^2$. (Hint: recall that $\sqrt{x^2 + y^2}$ is the distance from the origin to (x,y).)

   (b) Check your figure with that shown here. Then complete the following statements.

   If m_1 is the slope of $\overline{AC}$, then $m_1 =$ ___?___ .

   If m_2 is the slope of $\overline{BC}$, then $m_2 =$ ___?___ .



(c) $m_1 \cdot m_2 = \underline{\ ?\ } \cdot \underline{\ ?\ }$

$$= \frac{y^2}{(x + r)(x - r)} = \frac{?}{x(x - r) + r(\underline{\ ?\ })}$$

$$= \frac{y^2}{x^2 - xr + xr - r^2} = \frac{y^2}{(\underline{\ ?\ })}$$

(d) If $x^2 + y^2 = r^2$, then $y^2 = \underline{\ ?\ }$ and $x^2 - r^2 = \underline{\ ?\ }$.

Thus $m_1 \cdot m_2 = \frac{y^2}{-y^2} = \underline{\ ?\ }$.

Hence $\overline{AC} \perp \underline{\ ?\ }$.

Let us see what we have proved in Exercises 17-4b.

We began with $AO = OB = OC$. Thus we know that A, B, and C are points on the circle whose center is O and whose diameter is $\overline{AB}$. This can be described as saying that $\angle ACB$ is <u>inscribed in a semicircle</u>.

We proved that $\overline{AC} \perp \overline{BC}$, hence $\angle ACB$ is a <u>right angle</u>.

Since C could be <u>any</u> point on the circle, except A or B, we can state what we have proved as the general theorem:

<u>An angle inscribed in a semicircle is a right angle.</u>

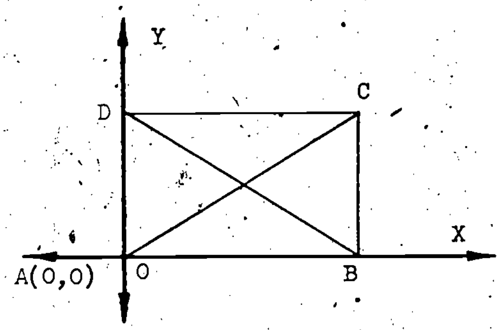
This is another way of stating what was proved in Chapter 15, Exercises 15-4b:

"Any angle formed by a vertex on a circle and by rays from the vertex through the endpoints of a diameter of that circle is a right angle."

Exercises 17-4c

Use coordinate geometry to prove each of the following.

1. The diagonals of a rectangle have equal lengths. (<u>Hint</u>: Place the axes as shown and finish assigning coordinates to the vertices of the rectangle.)



2. The midpoint of the hypotenuse of a right triangle is equidistant from its three vertices. (<u>Hint</u>: Locate the axes along the legs of the triangle and name the vertices of the acute angles $(2a,0)$ and $(0,2b)$.)

3. The diagonals of a square are perpendicular to each other.

4. The line joining the midpoints of two sides of a triangle is parallel to the third side of the triangle and equal to one-half of it. (<u>Hint</u>: Put one vertex at the origin, and the other two at $(2a,0)$ and $(2b,2c)$.)

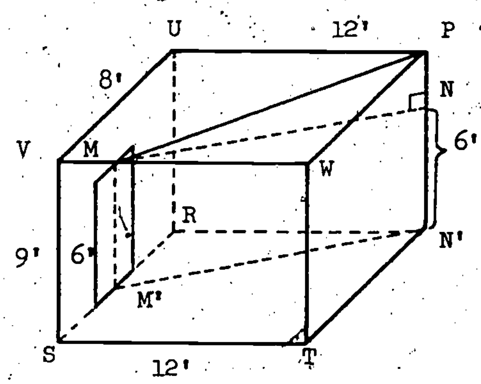
5. Every point on the perpendicular bisector of a segment is equidistant from the ends of the segment. (<u>Hint</u>: Put the segment along the X-axis, with the Y-axis the perpendicular bisector of the segment.)

6. $\triangle ABC$ is placed on a coordinate system with A at $(0,0)$, B at $(2b,2c)$ and C at $(2a,0)$. Through the midpoint M of $\overline{BC}$, $\overline{AM}$ is extended so that $\overline{MP} \cong \overline{AM}$.

   (a) Prove that ACPB is a parallelogram.

   (b) State what you have proved as a general theorem about extending a median of a triangle.

17-5. Three Dimensional Coordinate Systems



The drawing above represents a small room, 8 feet wide, 12 feet long and 9 feet high. There is a door 6 feet high in the center of one end. From point M, the midpoint of the top of the door frame, a wire is stretched to a top corner P, of the opposite wall. How long is the wire?

Exercises 17-5a

(Class Discussion)

1. Select N on PN' so that $N'N = 6$ ft. Thus $MM'N'N$ is a rectangle, and $\triangle MNP$ is a right triangle. $PN = \underline{\ ?\ }$.

2. By the Pythagorean Property, $PM = \sqrt{(PN)^2 + \underline{\ ?\ }}$; we do not know the length of $\overline{MN}$, but we do know that $\overline{MN} \cong \underline{\ ?\ }$.

3. $\triangle M'RN'$ is also a ___?___ triangle; $M'N' = \sqrt{(M'R)^2 + \underline{\ ?\ }}$.

4. $RN' = \underline{\ ?\ }$; M' is the midpoint of $\overline{RS}$, hence $M'R = \underline{\ ?\ }$.

   $M'N' = \sqrt{12^2 + \underline{\ ?\ }} = \sqrt{\underline{\ ?\ }}$.

5. $PM = \sqrt{\underline{\ ?\ } + 160} = \sqrt{\underline{\ ?\ }} = \underline{\ ?\ }$.

6. The wire is ___?___ feet long.

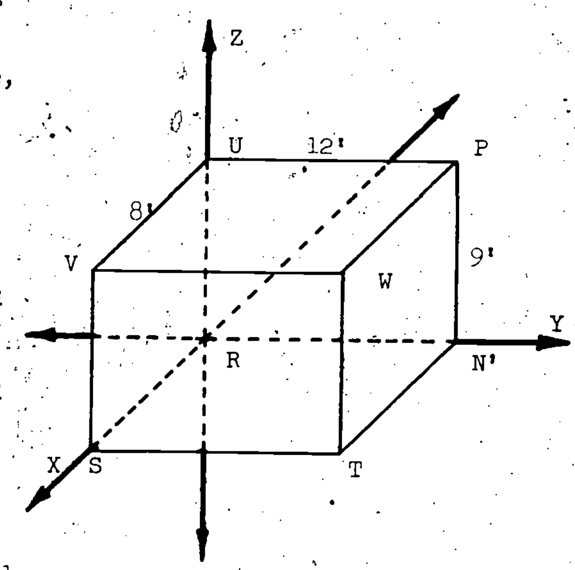
In the exercises above, we were concerned with finding a distance between two points in space, given numerical information which did not all relate to a single plane. This involves the location of a point in space by means of three numbers.

Given a point, we can describe its location <u>on a line containing the point</u> by assigning a coordinate system to the line and stating the unique number which is the coordinate of the point.

Given a point, we can describe its location <u>in a plane containing the point</u> by assigning two coordinate axes to the plane and stating the unique ordered pair of numbers which are the coordinates of the point.

If we need to describe the location of the given point <u>in space</u>, we need a third axis, perpendicular to the X- and Y-axes. We describe the point by stating an <u>ordered triple</u> of numbers.

For example, to a sketch of the room described at the beginning of this section, we might assign a coordinate system as indicated here, with the origin at point R. We shall assume that the positive direction of each axis is as follows:



X-axis: positive numbers <u>to the front</u>.

Y-axis: positive numbers <u>to the right</u>.

Z-axis: positive numbers <u>up</u>.

Now we can describe each point in the sketch above by an ordered triple (x,y,z). For example,

R is the point (0,0,0)

S is the point (8,0,0)

T is the point (8,12,0).

Exercises 17-5b

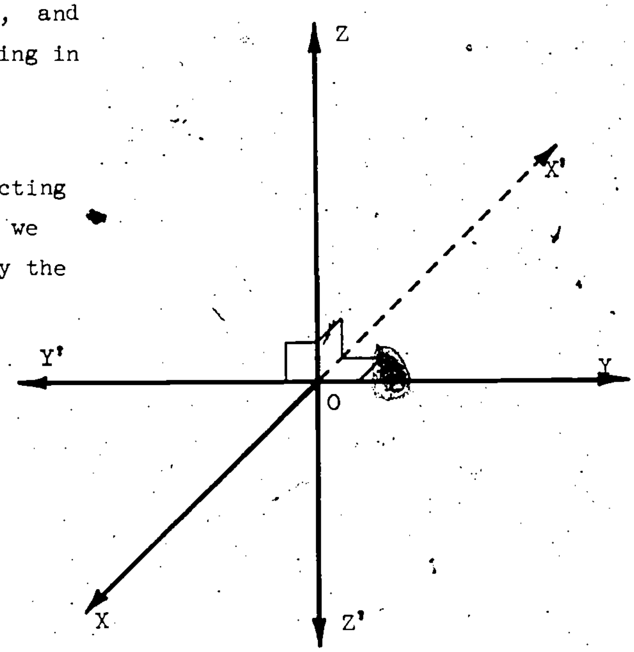
1. Give the ordered triples corresponding to points U, V, W, P, and N' in the figure above.

2. Give also the ordered triples corresponding to points M, M', and N in the figure at the beginning of the chapter, assuming the same assignment of axes as the one described.

3. Suppose that the origin had been placed at S instead of R, with the same agreement about the positive direction. Give the ordered triples with reference to the new axes for points M, M', N, N', P, R, S, T, U, V, and W.

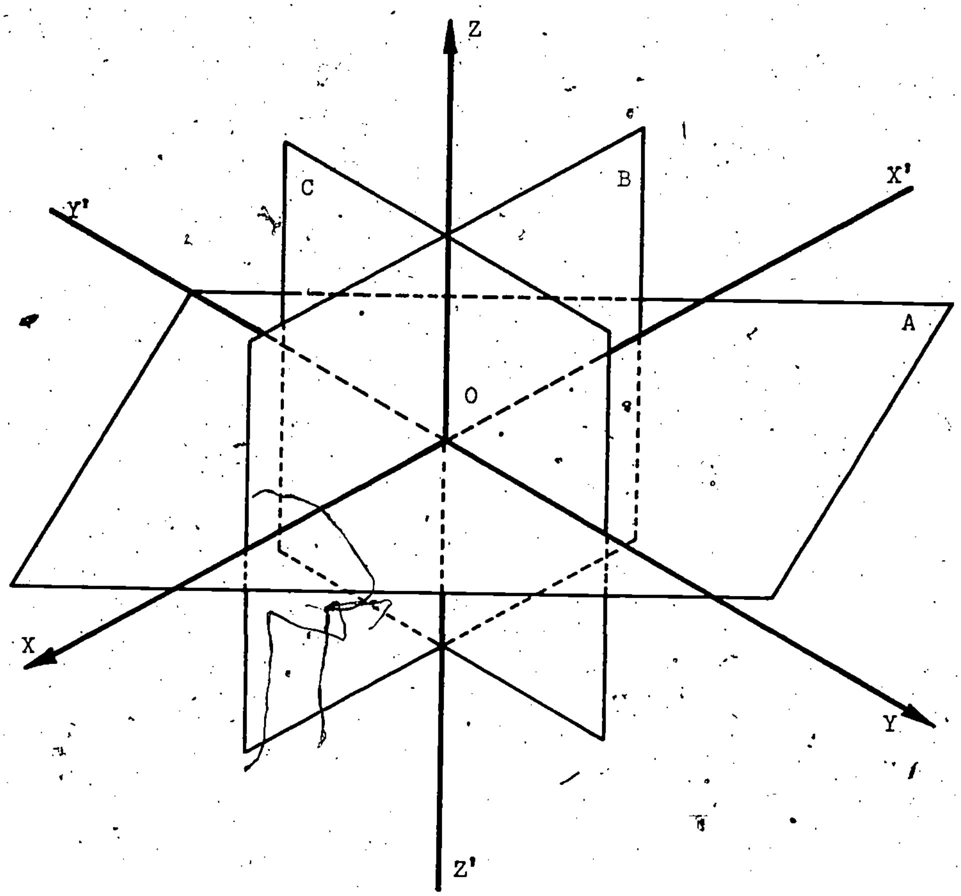
Suppose that $\overleftrightarrow{XX'}$, $\overleftrightarrow{YY'}$, and $\overleftrightarrow{ZZ'}$ are three lines intersecting in point O, with $\overleftrightarrow{XX'} \perp \overleftrightarrow{YY'}$, $\overleftrightarrow{YY'} \perp \overleftrightarrow{ZZ'}$, and $\overleftrightarrow{ZZ'} \perp \overleftrightarrow{XX'}$.

Recall that two intersecting lines determine a plane. Thus we have three planes determined by the three lines, taken in pairs.



Exercises 17-5c

The figure shown here pictures three planes A, B, and C, which are mutually perpendicular ($A \perp B$, $A \perp C$, $B \perp C$), with lines of intersection as indicated.



1. Of planes A, B, and C, which plane is determined by the intersecting lines

   (a) $\overleftrightarrow{XX'}$ and $\overleftrightarrow{YY'}$?

   (b) $\overleftrightarrow{XX'}$ and $\overleftrightarrow{ZZ'}$?

   (c) $\overleftrightarrow{YY'}$ and $\overleftrightarrow{ZZ'}$?

2. (a) $\overleftrightarrow{XX'}$ is the intersection of plane A with plane ___?___.

(b) The intersection of planes B and C is ___?___.

(c) Planes A and C intersect in ___?___.

(d) The intersection of all three planes is ___?___.

(e) The three planes divide space into ___?___ regions.

3. Planes A, B, and C can be named by identifying the axes which determine the planes. For example, plane A is the XY-plane. Give similar names for planes B and C.

4. Since planes A, B, and C are mutually perpendicular, their lines of intersection $\overleftrightarrow{XX'}$, $\overleftrightarrow{YY'}$, and $\overleftrightarrow{ZZ'}$ are also mutually perpendicular. Thus, they are a set of coordinate axes to which we can relate points in space. For any point $P(x,y,z)$ on $\overleftrightarrow{XX'}$, both y and z have the value 0, hence the coordinates of the point take the form $(x,0,0)$ where x is some real number. Express the form of the coordinates of P for each of the following cases.

(a) P is on $\overleftrightarrow{YY'}$.

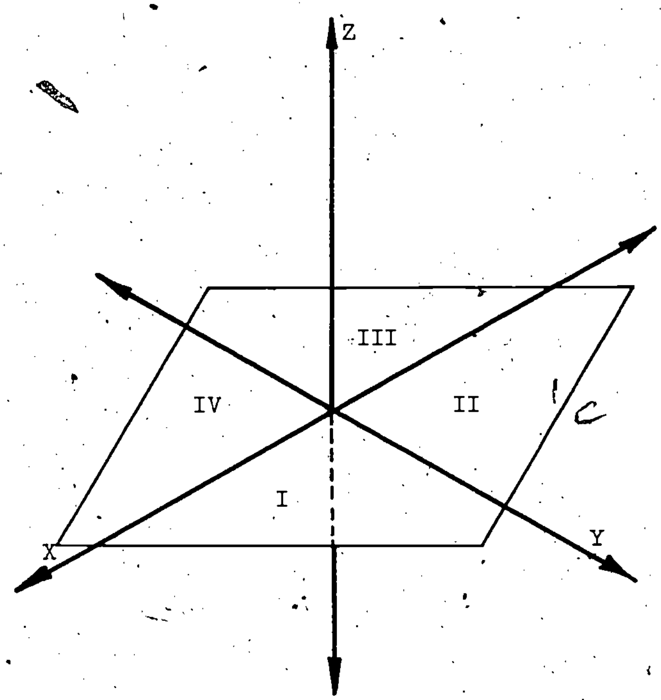
(b) P is on $\overleftrightarrow{ZZ'}$.

(c) P is in plane A.

(d) P is in plane B.

(e) P is in the YZ-plane.

5. The eight regions into which the three planes divide space are called <u>octants</u>. We shall number the upper four of these as indicated in the drawing on the next page, with octants I - IV corresponding to quadrants I - IV in plane A. The octants below plane A will be named in the same order, with V below I, VI below II, VII below III, and VIII below IV. For any point in octant I, all three coordinates are positive. In octant II, x is negative, y and z are positive. List the signs of the coordinates in the other six octants.



6. We shall assume that by using three mutually perpendicular axes, we can set up a one-to-one correspondence between the set of all ordered triples of real numbers and the set of points in space.

   (a) If a point is on one of the axes, at least ___?___ (how many) of its coordinates is (are) 0.

   (b) If a point is in one of the planes determined by a pair of axes, at least ___?___ of its coordinates is (are) 0.

   (c) If a point is in one of the eight octants, then ___?___ of its coordinates is (are) 0.

7. For each ordered triple below, describe where the point lies in relation to the coordinate system.

   Examples: (0,-2,0) is on the Y-axis.
   (5,0,-2) is in the XZ-plane.
   (-3,5,-1) is in octant VI.

   (a) (0,0,3)
   (b) (-2,3,0)
   (c) (0,3,-3)
   (d) (5,-3,-1)
   (e) (-5,-3,1)
   (f) (5,3,-1)

17-6. Distance Between Points in Space

We should recall here the distance formulas that we have found for two points on a line and for two points on a plane.

Exercises 17-6a

(Class Discussion)

1. If P_1 and P_2 are two points on the number-line, with coordinates x_1 and x_2, respectively, then $P_1P_2 = \underline{\ ?\ }$. Since $\sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$ we could write $P_1P_2 = \underline{\ ?\ }$.

2. For points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, two points in the number plane, $P_1P_2 = \underline{\ ?\ }$. If both points are on the x-axis, $y_1 = y_2 = \underline{\ ?\ }$, so the distance becomes $\underline{\ ?\ }$. If both are on the y-axis, then $P_1P_2 = \underline{\ ?\ }$.

3. Now think about two points in space, $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

   (a) If both points are on the same coordinate axis, at least two numbers in each ordered triple have the value $\underline{\ ?\ }$. For example, if both are on the Z-axis, $P_1 = (0,0,z_1)$ and $P_2 = (0,0,\underline{\ ?\ })$, hence $P_1P_2 = \sqrt{(z_2 - z_1)^2}$ or $|z_2 - z_1|$.

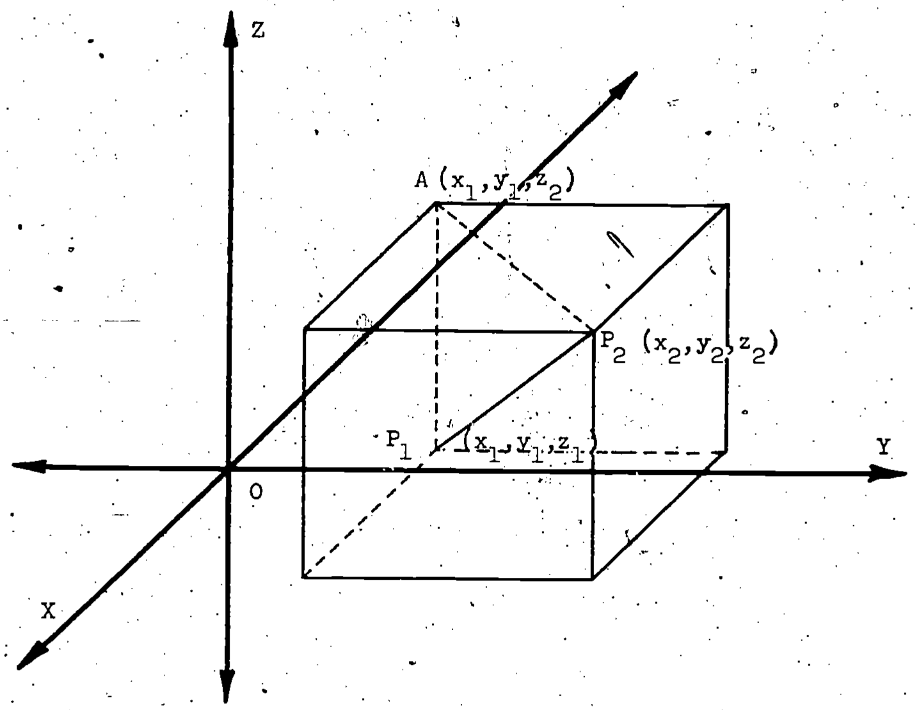
   (b) If both points are in the same coordinate plane, at least one number in each ordered triple is 0; for example, if both are in the XZ-plane, then both y_1 and $\underline{\ ?\ }$ are 0, so $P_1P_2 = \underline{\ ?\ }$.

   (c) Following this pattern:

Points		Distance P_1P_2
On a line	$P_1(x_1)$, $P_2(x_2)$	$\sqrt{(x_2 - x_1)^2}$
In a plane	$P_1(x_1,y_1)$, $P_2(x_2,y_2)$	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

   What do you guess that the formula would be for the distance between $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$?

In order to verify our guess concerning the formula for P_1P_2, given $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, let us look at this figure,



which we can imagine as a box with P_1 and P_2 at diagonally opposite corners, and with each edge of the box parallel to one of the coordinate axes.

From geometry we can show that, since $\overline{P_1A}$ is perpendicular to two lines in a plane parallel to the XY-plane, $\overline{P_1A}$ is perpendicular to any other line through A and lying in the plane. Specifically, $\overline{P_1A} \perp \overline{AP_2}$, hence $\triangle AP_1P_2$ is a right triangle, and $\overline{P_1P_2}$ is the hypotenuse of the triangle.

$\overline{AP_1}$ is parallel to the Z-axis, so we use the formula for distance on a line to compute

$$(AP_1)^2 = (z_2 - z_1)^2.$$

$\overline{AP_2}$ lies in a plane parallel to the XY-plane, so we use the formula for distance in a plane to compute

$$(AP_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

For the right triangle AP_1P_2, using the Pythagorean Property, we write

$$(P_1P_2)^2 = (AP_1)^2 + (AP_2)^2$$

$$= (z_2 - z_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

<u>Exercises 17-6b</u>

1. Find AB, given:

   (a) $A(4,-1,-5)$; $B(7,3,7)$

   (b) $A(0,4,5)$; $B(-6,2,8)$

   (c) $A(3,0,7)$; $B(-1,3,7)$

   (d) $A(3,4,5)$; $B(8,4,1)$

   (e) $A(0,1,0)$; $B(-1,-1,-2)$

   (f) $A(1,2,3)$; $B(0,0,0)$

2. Show that $\triangle ABC$ with vertices $A(2,4,1)$, $B(1,2,-2)$, $C(5,0,-2)$ is a right triangle.

3. Is the triangle with vertices $A(2,0,8)$, $B(8,-4,6)$, $C(-4,-2,4)$ isosceles? Justify your answer.

4. Determine if $\triangle ABC$ is equilateral, given:

   (a) $A(1,3,3)$, $B(2,2,1)$, $C(3,4,2)$

   (b) $A(6,2,3)$, $B(1,-3,2)$, $C(0,-2,-5)$

5. Show that the opposite sides of the figure $ABCD$ with vertices $A(3,2,5)$, $B(1,1,1)$, $C(4,0,3)$, $D(6,1,7)$ are congruent.

6. Given $P_1(2,-3,5)$ and $P_2(5,3,-1)$.

   (a) Find P_1P_2.

   (b) The axes are translated so that the origin moves to the point whose coordinates are $(2,0,0)$. What are the new coordinates of P_1 and P_2?

   (c) The axes are moved again so that the origin is at the point whose original coordinates were $(2,-3,0)$. What coordinates do P_1 and P_2 now have?

   (d) A final translation of the axes takes the origin to $(2,-3,5)$. Give the coordinates of P_1 and P_2 under this transformation.

For $\overline{P_1P_2}$, we have formulas for finding the coordinates of its midpoint M, if P_1 and P_2 are on the number line or are in the coordinate plane.

Exercises 17-6c

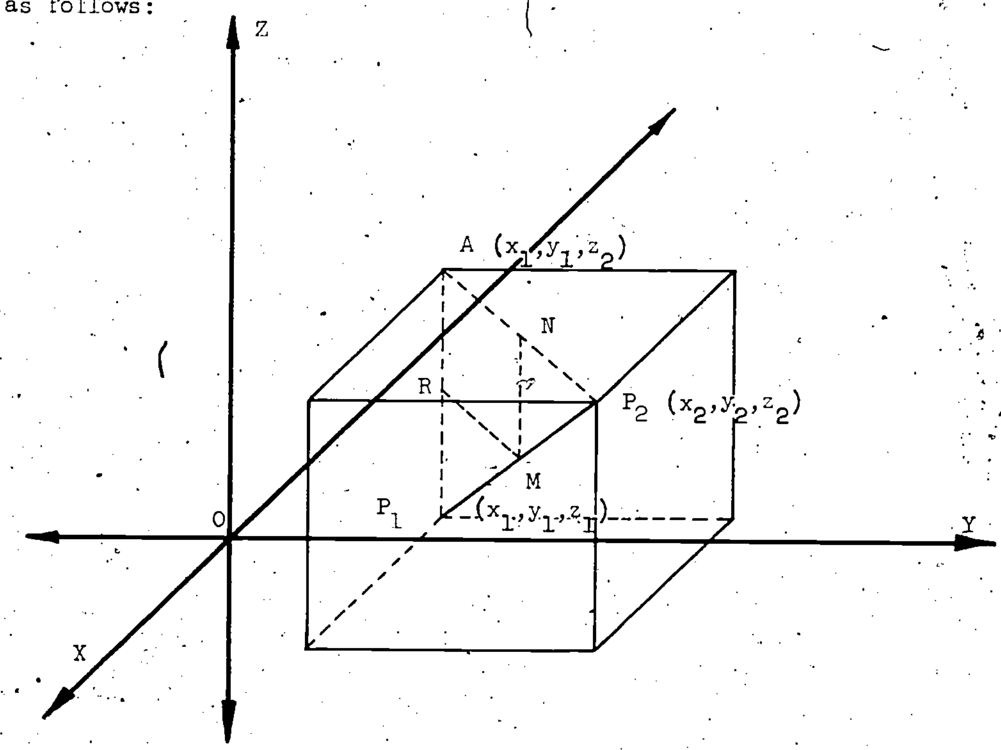
(Class Discussion)

1. Given $P_1(x_1)$ and $P_2(x_2)$, two points <u>on the number line</u>, the coordinate of M is __?__.

2. Given $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, two points <u>in the coordinate plane</u>, the coordinates of M are given by the ordered pair (__?__, __?__).

3. What do you guess the coordinates of M would be if $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ are thought of as two points <u>in space</u>?

The fact that the coordinates of M are, as you probably guessed, $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$ might be proved, using the figure shown here, as follows:



Assume that M is the midpoint of $\overline{P_1P_2}$. Draw $\overline{MN} \parallel \overline{AP_1}$ and $\overline{MR} \parallel \overline{AP_2}$, as shown, with N on $\overline{AP_2}$ and R on $\overline{AP_1}$. Then N is the midpoint of segment $\overline{AP_2}$ in the "Z_2-plane", which is parallel to the XY-plane, and R is the midpoint of $\overline{AP_1}$, a segment parallel to the Z-axis.

N has xy-coordinates $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$, and since $\overline{MN} \perp \overline{AP_2}$ at N, these are also the xy-coordinates of M.

R has a z-coordinate $\frac{z_1 + z_2}{2}$; which is also the z-coordinate of M, since $\overline{MR} \perp \overline{AP_1}$.

Thus M is the point $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2})$.

Exercises 17-6d

1. For each of the following, find the length of $\overline{AB}$ and the coordinates of the midpoint of $\overline{AB}$.

   (a) A(0,4,5); B(-6,2,8)

   (b) A(3,0,7); B(-1,3,7)

   (c) A(0,0,0); B(-4,6,-3)

2. The coordinates of the midpoint of a segment are (8,-4,1). If the coordinates of one end point of the segment are (4,-1,3), find the coordinates of the other end point.

3. (a) For each of the following find the lengths of the sides of the triangle:

   $\triangle$ ABC with A(5,9,11); B(0,-1,-4); C(5,-11,1)

   $\triangle$ DEF with D(4,3,-4); E(-2,9,-4); F(-2,3,2)

   $\triangle$ GHJ with G(2,4,2); H(4,5,4); J(4,2,1)

   (b) For each triangle in part (a), answer each of these questions:

   Is it a right triangle?

   Is it an isosceles triangle?

   Is it an equilateral triangle?

4. Recall that if $AB + BC = AC$, then A, B, and C are collinear points, with B between A and C. For each of the following decide whether or not the points are collinear and if they are, which two are the end points.

(a) $A(3,-2,-7)$; $B(6,4,-5)$; $C(12,16,-1)$

(b) $A(1,2,5)$; $B(5,-10,-1)$; $C(3,-4,2)$

(c) $A(0,4,3)$; $B(1,6,1)$; $C(-1,5,3)$

(d) $A(1,2,4)$; $B(7,-7,22)$; $C(-1,5,-2)$

5. In Exercises 17-6b you showed that if the vertices of the figure $ABCD$ are $A(3,2,5)$; $B(1,1,1)$, $C(4,0,3)$, and $D(6,1,7)$, then the opposite sides are congruent. Now decide whether A, B, C, D are coplanar by finding the midpoints of the diagonals $\overline{AC}$ and $\overline{BD}$. Explain how this helps you to decide.

17-7. Algebraic Description of Subsets of Space

The use of an ordered triple to describe a point by stating its position with respect to three coordinate planes can literally add a new dimension to our thinking. For example, consider the set of points for which the sentence "$x - 1$" is true. Before we can describe the points, we must determine the larger set from which our set of points is to be selected. Is it a line? or a plane? or all of the points in space?

Exercises 17-7a

(Class Discussion)

1. Describe these two sets of points:

(a) $\{x : x = 1\}$

(b) $\{(x,y) : x = 1\}$

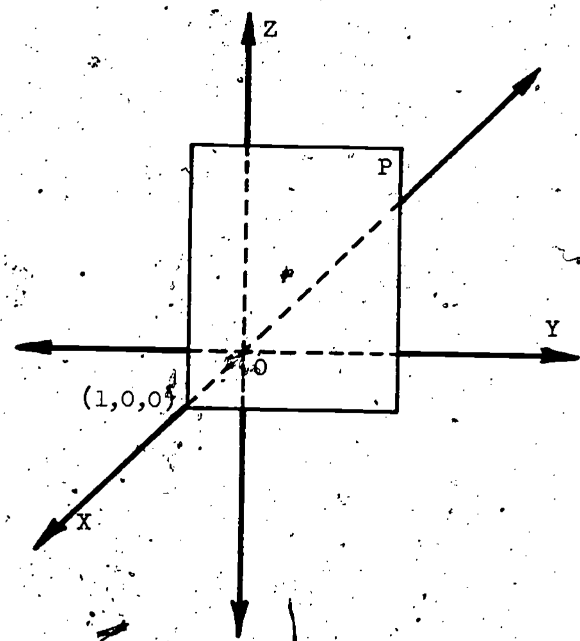
2. Consider this set:

$\{(x,y,z) : x = 1\}$.

(a) Here we are looking for all of the points in space for which the first coordinate is ___?___.

(b) The points we want will all lie in the same __?__.

(c) The plane containing the points is parallel to the __?__ coordinate plane, as represented by plane P in the sketch.



3. Describe the set of points 3 inches from the origin.

(a) If we mean the set of points <u>on the number line</u> whose distance from the origin is 3 inches, we can describe the set by the notation $\{-3,3\}$, or by $\{x : |x| = 3\}$ or by

$$\{x : x^2 = \underline{\ \ ?\ \ }\}.$$

The graph of this set consists of __?__ point(s).

(b) If we mean the set of points <u>in the coordinate plane</u> whose distance from the origin is 3, we recall the distance formula, $\sqrt{(x_2 - x_1)^2 + \underline{\ \ ?\ \ }}$. Since one point is the origin, we can write

$$d = \sqrt{\underline{\ \ ?\ \ } + y^2}.$$

or

$$d^2 = \underline{\ \ ?\ \ } + \underline{\ \ ?\ \ }.$$

For the set of points for which the distance is 3 we could write

$$\{(x,y) : x^2 + \underline{\ \ ?\ \ } = \underline{\ \ ?\ \ }\}$$

The graph of this set is a circle with radius ___?___ and center at the ___?___.

(c) If we mean the set of points <u>in space</u> 3 inches from the origin we write

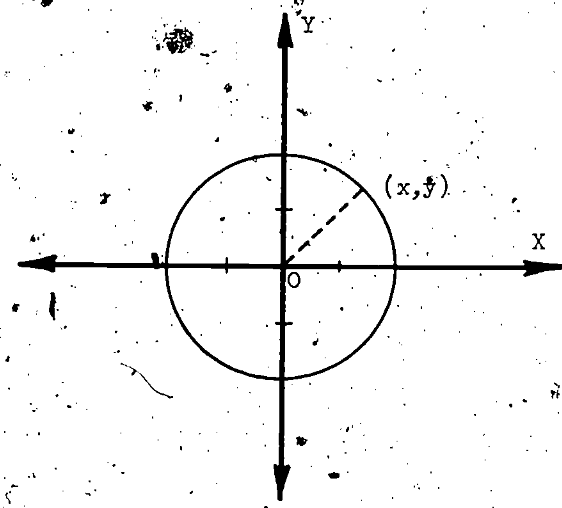
$$\{(x,y,z) : \underline{\ ?\ } + \underline{\ ?\ } + \underline{\ ?\ } = 9\}$$

The graph of this set is a ___?___, with radius ___?___ and center at the origin.

4. Describe each of the following sets as definitely as possible:

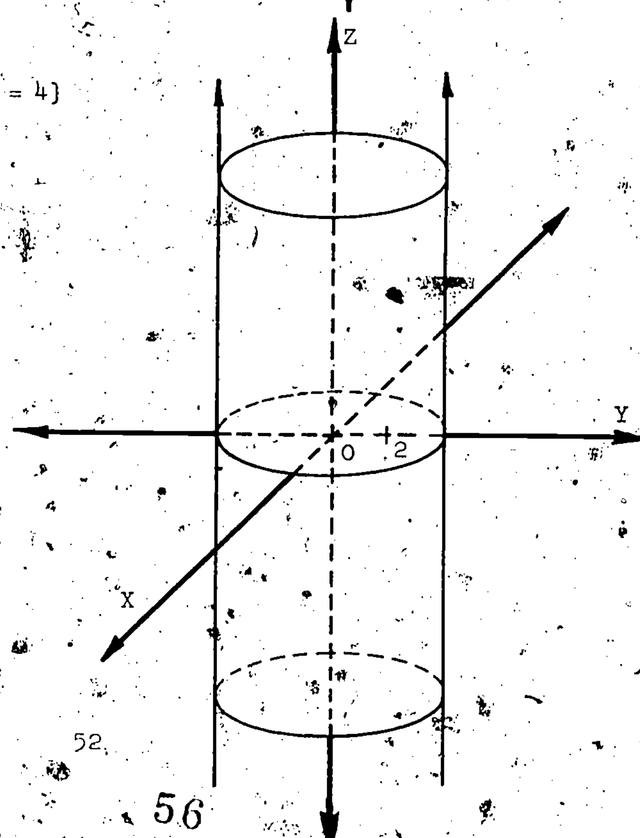
(a) $\{(x,y) : x^2 + y^2 = 4\}$

The set consists of all points in the coordinate ___?___ whose distance from the origin is ___?___. The graph of the set is a ___?___ with center at the ___?___, as shown in the sketch.



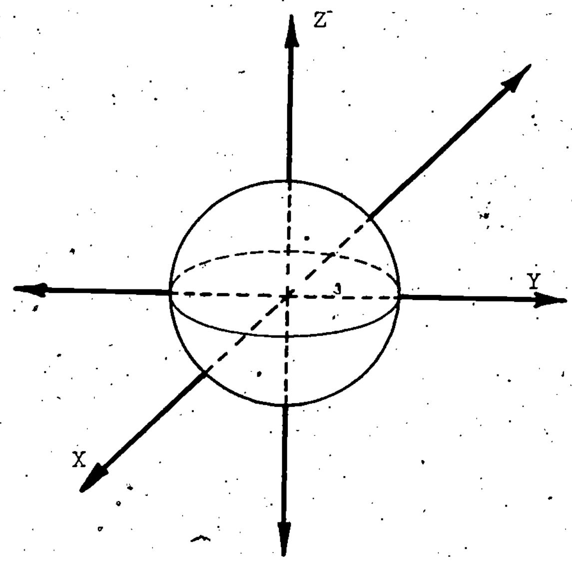
(b) $\{(x,y,z) : x^2 + y^2 = 4\}$

The set consists of all points in ___?___ whose distance from the z-axis is ___?___. The graph is a cylindrical surface perpendicular to the ___?___-plane whose intersection with that plane is a ___?___ of radius 2, as shown in the sketch.



(c) $\{(x,y,z) : x^2 + y^2 + z^2 = 4\}$

The set consists of all points in __?__ whose distance from the __?__ is 2. The graph is a __?__ of radius __?__ with center at the __?__.



Exercises 17-7b

Describe each of the following sets as definitely as possible; draw graphs as directed.

1. (a) $\{x : x = 3\}$

   (b) $\{(x,y) : x = 3\}$

   (c) $\{(x,y) : x + y = 3\}$

2. (a) $\{(x,y,z) : x = 3\}$

   (b) $\{(x,y,z) : x + y = 3\}$

   (c) $\{(x,y,z) : x + y + z = 3\}$

3. (a) $\{x : x^2 = 2\}$

   (b) $\{(x,y) : x^2 = 2\}$

   (c) $\{(x,y,z) : x^2 = 2\}$

4. Describe each and draw the graph.

(a) $\{(x,y) : x^2 + y^2 = 1\}$

(b) $\{(x,y,z) : x^2 + y^2 = 1\}$

(c) $\{(x,y,z) : x^2 + y^2 + z^2 = 1\}$

5. Describe each and draw the graph.

(a) $\{x : x > 5\}$

(b) $\{(x,y) : x > 5\}$

(c) $\{(x,y,z) : x > 5\}$

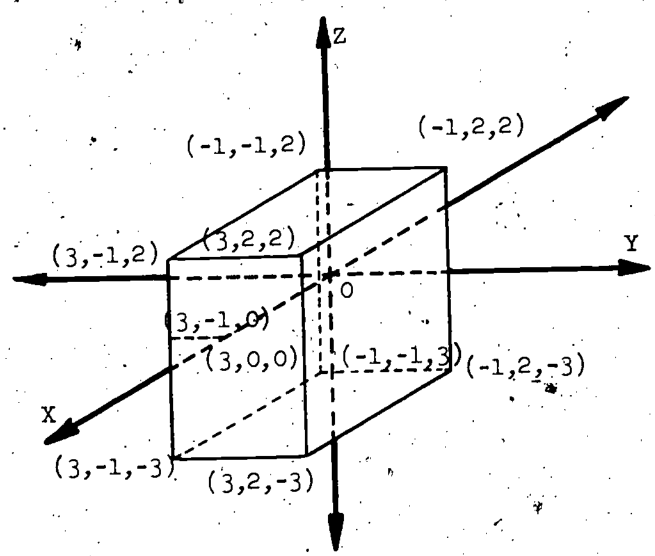
6. Describe each and draw the graph for (a) and for (b).

(a) $\{x : -1 \leq x \leq 3\}$

(b) $\{(x,y) : -1 \leq x \leq 3 \text{ and } -1 \leq y \leq 2\}$

(c) $\{(x,y,z) : -1 \leq x \leq 3 \text{ and } -1 \leq y \leq 2 \text{ and } -3 \leq z \leq 2\}$.

The graph for (c) is:



17-8. Summary

Section 17-1.

The slope of a nonvertical line is the value of m in the equation of the line, written in the form $y = mx + b$.

The slope of any horizontal line is 0.

The equation of a vertical line cannot be written in the form $y = mx + b$. No slope is assigned to a vertical line.

If two points on a nonvertical line are known, the number which is the slope of the line is given by the fraction

$$\frac{\text{difference of ordinates}}{\text{difference of abscissas}} .$$

If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two points on a nonvertical line, then the slope m of $\overleftrightarrow{P_1P_2}$ is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} .$$

If $m < 0$ the line slopes downward from left to right; if $m = 0$ the line is horizontal; if $m > 0$, the line rises from left to right.

If the slope of a line and a single point on the line are known the line can be located very quickly. If the slope is positive, more points are located by moving "to the right and up"; for a negative slope, move "to the right and down".

The point where the line intersects the Y-axis is called the y-intersection of the line; the value of y at this point is called the y-intercept.

The form $y = mx + b$ is called the slope-intercept form of a linear equation because the number m is the slope of the line and the number b is its y-intercept.

Section 17-2.

An equation of a line can be written, given the slope m and the coordinates (x,y_1) of a point, as follows:

(1) Substitute m, x_1, and y_1 in the form $y = mx + b$, then

(2) solve the resulting equation for b, then

(3) substitute m and b in $y = mx + b$.

If two points are known, the slope can be computed from the coordinates of the points; then an equation can be written as described above.

An alternate way of writing an equation of the line through $P(x_1,y_1)$ with slope m is

$$y - y_1 = m(x - x_1).$$

Section 17-3.

The following theorems are stated and proved:

Theorem 17-1. The graph of any equation of the form $y = mx + b$ is a straight line.

Theorem 17-2. Two nonvertical lines are parallel if and only if they have the same slope.

Theorem 17-3. Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

Section 17-4.

A shift of either or both of the coordinate axes such that the X-axis remains horizontal and the Y-axis remains vertical is called a translation of the axes.

A property is said to be invariant under a translation if the property is the same after the translation as it was before.

Invariant properties under translation of axes include

(1) distance between two points,

(2) slope of line,

(3) parallelism of lines,

(4) perpendicularity of lines.

Coordinate proofs of many geometric facts are possible, based on the Distance Formula, the Midpoint Formula, and the three theorems listed in Section 17-3.

In writing a coordinate proof, we begin by

(1) placing coordinate axes in the plane of the figure in a way that makes it easy to assign coordinates to some points in the figure, and

(2) using known properties of the figure to assign coordinates to other points in terms of the coordinates already assigned.

Section 17-5.

For points in space, a coordinate system consists of three coordinate axes, mutually perpendicular; a point is described by an ordered triple of numbers.

The three coordinate axes, taken in pairs, determine three mutually perpendicular planes.

Section 17-6.

If $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ are two points in space, then

(1) the length of $\overline{P_1P_2}$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$,

(2) the midpoint of $\overline{P_1P_2}$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Section 17-7.

Before we can describe the set of points for which a sentence containing less than three variables is true, we need to know the larger set from which our set of points is to be selected.

For example,

$\{x : x = a\}$ is a point,

$\{(x,y) : x = a\}$ is a line,

$\{(x,y,z) : x = a\}$ is a plane.

Chapter 18

PROBLEM ANALYSIS

18-1. Some "Mysterious Mathematical Powers"

How do people make discoveries? Is there some mysterious process that leads them to the solution of problems in mathematics, nuclear fission, rocket design, conservation of natural resources, and the like? As you may know, there are no simple answers to the above questions. However there are some approaches to problem analysis which can be helpful in organizing your thinking, and which are easy to learn. We will discuss many different problem situations in order to illustrate and develop some strategies that can be used to help solve problems, but we will not be overly concerned with the complete solutions for all of these problems.

Some of the following situations may be surprising to you. We hope to illustrate the power that one can have when he knows how to translate verbal statements into symbolic statements. The techniques introduced here will be useful in the analysis of many different kinds of problems. If you learn how to analyze these situations, then you should be able to write directions which will "force" someone to reveal to you such things as his age, how much change he has in his pocket, his address, and the like. For example, using the following instructions, you can "force" him to get a particular number for an answer. Try it out for yourself.

(1) Choose any number, other than zero.

(2) Add three to the number.

(3) Multiply the sum by two.

(4) Subtract six from the product.

(5) Divide this difference by the original number.

Everybody's answer will be 2, regardless of his choice of the beginning number! Why is this true? Let us see if we can analyze the situation, and perhaps learn how to create a set of instructions which will give as an answer any number you may want.

Exercises 18-1a

(Class Discussion)

1. Suppose that we let N represent a number, $N \neq 0$.

   (a) Describe the output of the function $f : N \to N + 3$.

   (b) Describe the output of the function $g : N \to 2(N + 3)$.

   (c) Describe the output of the function $h : N \to 2(N + 3) - 6$.

   (d) Describe the output of the function $j : N \to \frac{2(N + 3) - 6}{N}$.

   (e) Simplify the expression which is the output of the function j.

   (f) If N is a number and $N \neq 0$, then what number will the expression in the output of j always represent? Do you see why the trick works?

2. Follow these instructions:

   (1) Multiply your age, in years, by four.

   (2) Add ten to the product of your age and four.

   (3) Multiply this sum by twenty-five.

   (4) Subtract the number of days in a year, (365), from the product.

   (5) Add the amount of change in your pocket (less than \$1.00) to this difference.

   (6) Add 115 to this sum.

The two digits on the right will represent the amount of change, and the remaining digits on the left will represent your age. If you translate the above instructions, as in Exercise 1, this translation will help you understand why this puzzle always works. Let A represent an age in years and C an amount of change less than \$1.00.

   (a) Write a series of functions, like those in Exercise 1, whose outputs can be described by the instructions listed above.

   (b) Simplify the expression in the output of the last function.

   (c) Did you get $100A + C$ when you simplified the expression in the output of the last function?

(d) Why is the ones digit of A in the hundreds place of the final answer? Why is C in the ones place or the ones and tens places of the final answer?

(e) If C was a dollar or more, then what effect would this have on the hundreds place of the final answer?

In Exercise 2 above, if you ask someone for his answer before adding 115, and then do this addition mentally, it will appear that you have "mystical" powers. Actually your powers will have a mathematical foundation, but this may not be apparent to your audience. However, we hope that you see how the process of "translation" helps you to understand the situation, and to see why the number "trick" works. Translation techniques are of considerable importance in the understanding and the organization of many problems, and we shall use them as a basic problem analysis strategy.

Exercises 18-1b

1. Show, using translation techniques, that by following the instructions below you can cast a "spell" upon a person. No matter what number he chooses, the result of following the instructions carefully, will always be the "lucky" number 7.

   (1) Choose a number.

   (2) Add three to the number.

   (3) Multiply this sum by two.

   (4) Add eight to this product.

   (5) Subtract twice the original number from this sum.

   (6) Divide this difference by 2.

2. Cast a "Halloween" spell on a friend by writing a set of four or more instructions which will always result in the "unlucky" number 13 as the final answer. Use translation techniques to show that your instructions will always work no matter what number he chooses.

3. If B represents the number of the birthday month, (1 for January, 2 for February, etc.), and D represents the number of the day of the month, then show that the following instructions will enable you

to determine a person's birthday.

(1) Add 7 to the number of the birthday month.

(2) Multiply this sum by 20.

(3) Subtract 40 from this product.

(4) Multiply this difference by 5.

(5) Add the day of the birthday month to this product.

Mentally subtract 500 from the last sum. The last two digits on the right will represent the day of the month and the digit or digits on the left of the numeral will represent the number of the month.

4. Pick any three consecutive numbers. Add them together. Divide that sum by three. Subtract one from this quotient. Show that the result will always be the first of the three consecutive numbers chosen.

5. Show why the following number "trick" always works: Write down your house number. (Omit any fractions.) Multiply it by four. Add twelve to this product. Multiply this sum by twenty-five. Add your age to this product. Subtract 365, the number of days in a year. Add 65. The tens and units digit of your answer will give you your age, and the other digits will represent your house number.

18-2. Translation of Phrases

Problems can occur in many different forms and they are not usually well understood at first. In fact, one significant activity in problem analysis is the process of determining what important questions are really involved in the problem situation. However, a major step toward the analysis, and eventual solution, of many problems is the translation of the problem into a convenient form which will permit some manner of organized analysis.

Many of the first phrases that you will be asked to translate will be quite simple. However, we will use the translation of these simple phrases to illustrate the skills that are helpful in the translation of complex sentences.

Some of the things that are useful to do when translating a phrase are: (1) to identify each variable as representing an unspecified number,

(2) to state, as completely as possible, what each variable represents (e.g., x represents the number of dollars in the cost of a bicycle),

(3) to identify any basic operations that are suggested in the phrase, and

(4) to identify all functions in the phrase and clearly describe the outputs of these functions.

In the following exercises you are asked to translate from mathematical phrases to English phrases and from English phrases to mathematical phrases. We hope that this process will enable you to see the frequent need for a detailed step-by-step analysis of problem situations.

Exercises 18-2a

(Class Discussion)

1. Given the function $f : (R, W) \to R - \frac{1}{4} W$. Translate the expression which is the output into an English phrase by completing the following steps:

   (a) In the output of function f, we will assume that R and W are names for $\underset{\text{(people, numbers)}}{\underline{\quad ? \quad}}$.

   (b) The phrase $\frac{1}{4} W$ represents the ___?___ of the numbers $\frac{1}{4}$ and W.

   (c) The phrase $R - \frac{1}{4} W$ represents the $\underset{\text{(sum, difference)}}{\underline{\quad ? \quad}}$ of the two numbers R and $\frac{1}{4} W$.

   (d) Usually we find that variables represent numbers with more meaning attached to them. If R represents a number of right answers a student got on a test, then can you describe W more completely? If possible, do so.

   (e) Write an English translation of the phrase which is the output of the function $f : (R, W) \to R - \frac{1}{4} W$ where R represents a number of correct answers a student got on a multiple choice test and W represents the number of wrong answers he got on the same test.

2. Find a function whose output is described by the following English phrase:

"For a typical student the total daily intake, in grams, of vitamin A and vitamin B, if the vitamin B intake is three times the vitamin A intake."

(a) If K represents a number of grams of vitamin A consumed by a typical student, then the output of the function

$$g : K \to 3K$$

can be described as ___?___.

(b) The word "total" here indicates the operation of addition. Therefore a function whose output could serve as a mathematical model of the English phrase stated for this problem is:

$$h : K \to K + \underline{\quad ? \quad}.$$

3. In the following, the word "translation" is used in the loose sense, meaning that each of the symbols for the operations of addition, subtraction, multiplication, and division, may lead to a variety of commonly used English phrases. For each example write two additional phrases that might lead to the indicated operation.

(a) The symbol "+", in a phrase like "a + b", is sometimes translated "the sum of".

(b) The symbol "-", in a phrase like "a - b", is sometimes translated "decreased by".

(c) The symbol "·", in a phrase like "a · b", is sometimes translated "the product of".

(d) The symbol "——", in a phrase like "$\frac{a}{b}$", is sometimes translated "the quotient of".

We are asking you to translate phrases in two directions. We intend this to be a first step toward the precise translation necessary,

(1) to construct a good mathematical model of a problem situation, and

(2) to interpret verbally your conclusions from that model.

Exercises 18-2b

Write an English phrase which describes the outputs mentioned in the functions below. State clearly what each variable represents in your translation. (Exercises 1 through 12.)

1. $f : t \to 60\,t$
2. $f : (L,W) \to 2L + 2W$
3. $f : d \to (3.14)\,d$
4. $f : (b,h) \to \frac{1}{2}\,bh$
5. $f : s \to s^2$
6. $f : (p,r,t) \to p \cdot r \cdot t$
7. $f : (d,r) \to \frac{d}{r}$, $r \neq 0$
8. $f : r \to (2)\,(3.1416)\,r$
9. $f : (\ell,w) \to \ell\,w + \ell\,w$
10. $f : (a,b,c) \to a + b + c$
11. $f : (x,y) \to x - 2y$
12. $f : m \to m - 5$

In each of the exercises determine a function whose output could be described by the given English phrase. Identify clearly what the given variable or variables represent. Choose a variable if none is given and use only one variable unless directed to do otherwise. (Exercises 13 through 18.)

13. The number of dollars earned in t hours at three dollars an hour.
14. The number of yards in t feet.
15. The average of three test scores x, y, and z.
16. The sum of two consecutive integers.
17. The product of two consecutive even integers.
18. Fifteen inches more than twice the number of inches in the length of a rectangle.

18-3. Translation of Sentences

You have found solution sets of sentences formed by combining mathematical phrases, such as "$3n + 1$", "2", "$25 - 7$", etc., with mathematical verb forms like "=", ">", "<". We find that many times one or two mathematical sentences can serve as a model of a problem situation which takes several English statements to describe.

Consider the following sentence:

$$35x + (70)(40) = 50(x + 40).$$

Translating this sentence into English is like writing a story when you know the ending. Let's see if we can work backward and write a problem which is an English translation of the sentence.

In the above sentence it is possible to identify several different functions. It is usually helpful to describe the outputs of these functions.

$f : x \to 35x$ $\qquad$ $h : x \to x + 40$

$g : x \to 35x + (70)(40)$ $\qquad$ $j : x \to 50(x + 40)$

The first thing we need to do in order to interpret the output of any of these functions is to agree upon what the variable x represents. Let's agree to the following statement as one interpretation of the variable x.

> "x represents a number of ounces of gold in a compound." (It is only fair to point out that we may have to revise and clarify this statement as we become more involved in the problem.)

We also need to agree upon what the numbers 35, 70, 40, and 50 represent. Suppose that:

35 represents the cost, in dollars, of one ounce of gold,

40 represents the number of ounces of platinum in the compound,

70 represents the cost, in dollars, of one ounce of platinum, and

50 represents the cost, in dollars, of one ounce of the mixture of gold and platinum.

Exercises 18-3a

(Class Discussion)

1. Interpret the output of the function $f : x \to 35x$.
2. What is the interpretation of $(70)(40)$?
3. Interpret the output of the function $g : x \to 35x + (70)(40)$.
4. Interpret the output of the function $h : x \to x + 40$.
5. Interpret the output of the function $j : x \to 50(x + 40)$.
6. Is the following a possible translation of the sentence?

   "The plating of a secret satellite must be a mixture of pure platinum and gold. Exactly forty ounces of pure platinum must be used in order for the satellite to perform correctly.

Pure platinum costs 70 dollars an ounce, and gold costs 35 dollars an ounce. The cost of the mixture of platinum and gold must be 50 dollars an ounce. How many ounces of gold must be used in order to make a compound which costs 50 dollars an ounce?"

7. Now try writing a translation of your own where we agree that x represents a number of pounds of lemon drops in a candy store mixture of lemon drops and caramels.

In the following exercises you will find that giving a clear description of the output of each indicated function will help you organize the English translation of the given mathematical sentence. In order to create a translation whose parts fit together in a reasonable way it is usually necessary to describe completely all of the variables and numbers indicated in the sentence. In describing the outputs of the indicated functional relationships, you will identify the various indicated operations, and interpret the effect of those operations on the numbers and variables involved. To be sure, many of the steps described above are eventually done mentally or are combined into a single step, but at this point we feel that you will gain some helpful skills and understandings by investigating this translation process in detail.

Example 1: Given the sentence $20 + \frac{y}{1500} = 24$, a translation could be made as follows:

Let 20 represent the number of times per minute that a man breathes at sea level,
and y represent a number of feet a man is above sea level,
and 1500 represent the number of feet of increase in altitude that corresponds to an increase of one breath per minute.

Indicated Functional Relationships	Descriptions of Outputs
$f : y \rightarrow \frac{y}{1500}$	"the number of extra breaths per minute a man will take at an altitude of y feet above sea level."

$g : y \rightarrow 20 + \frac{y}{1500}$ "the total number of breaths per minute a man will take at an altitude of y feet."

$h : y \rightarrow 24$ "the total number of breaths per minute a man will take at an altitude of y feet."

Note that the outputs of function g and function h must be described in exactly the same way since the sentence, $20 + \frac{y}{1500} = 24$, states that the outputs of functions g and h are equal.

A translation can now be written:

> "If a man breathes 20 times per minute at sea level and if he takes one extra breath per minute for each increase of 1500 feet in altitude, then how high above sea level will he be if he breathes 24 times a minute?"

Exercises 18-3b

Write English translations, in problem form, for the following sentences. State clearly your choice as to what each variable represents. Then state what each part of the sentence represents, and interpret the output of each function involved. Be sure that your translation makes sense! Be creative, use your imagination!

1. $24 = 12w$
2. $40 = 2b + 2(3b)$
3. $312 = (\frac{1}{2})(48)(n)$
4. $64 = a^2 + 25$
5. $\frac{1}{30} + \frac{1}{50} = \frac{1}{x}$
6. $x + (x + 1) + (x + 2) = 72$
7. $|y| = 5$
8. $\frac{75 + 65 + z}{3} = 77$

In many situations we also want to translate English sentences into mathematical sentences. We find that essentially the same things must be done to translate from English sentences to mathematical sentences as was done in the translation of mathematical sentences to English sentences. It is useful:

(1) to state clearly and completely what the variable or variables represent;

(2) to represent carefully the output of each of the functions involved in the problem situation, and to interpret each output. (Note that this involves recognizing the basic operations implied in the problem statement.);

(3) to identify mathematical verb forms that are implied in the problem statement; and

(4) to be sure that your translation makes sense.

The following examples illustrate the process of translating English sentences to mathematical sentences. Read through each of the steps carefully.

Example 2: "The distance an object falls during the first second is 32 feet less than the distance it falls during the second second. During the two seconds it falls 48 feet or less, depending on the air resistance. How far does it fall during the second second?"

(1) d represents a number of feet an object falls during the second second.

(2) $f : d \to d - 32$ is a function the output of which represents a distance an object falls during the first second.

(3) $g : d \to (d - 32) + d$ is a function the output of which represents a total distance an object falls during two seconds.

(4) 48 represents the maximum number of feet an object falls during two seconds.

(5) The phrase "48 feet or less" implies the verb form "$\leq$".

(6) The mathematical sentence could be:

$$(d - 32) + d \leq 48.$$

Check Your Reading

1. What part of the written problem leads to the function $f : d \to d - 32$?

2. What part of the written problem leads to the function $g : d \to (d - 32) + d$?

3. What part of the written problem supports the use of the word "maximum" in describing the role of the number "48" in the problem?

4. Is there any restriction on the replacement set of the variable that is caused by physical restrictions in the problem situation?

Sometimes one of the most difficult parts of the translation process is discovering the functional relationships that are involved in the problem. The following is one approach that may help you in this task.

Example 3: "In a gasoline economy test, one driver, starting with the first group of cars, drove for 5 hours at a certain speed and was then 120 miles from the finish line. Another driver, who set out later from the same place with a second group, had traveled at the same rate as the first driver for 3 hours and was 250 miles from the same finish line. How fast were these two men driving?"

Let r represent a speed of a car, measured in miles per hour.

If you choose a specific input value for r, say 50, this will sometimes help you see some of the functional relationships involved in the problem situation. The resulting arithmetic phrases, the description of those phrases, and the implied functional relationships are listed below.

Arithmetic Phrase	Description of Phrase	Function
(5)(50)	The distance a car goes in five hours at 50 miles per hour.	$f : r \rightarrow 5r$
(3)(50)	The distance a car goes in three hours at 50 miles per hour.	$g : r \rightarrow 3r$
(5)(50) + 120	120 miles more than the distance a car goes in 5 hours at 50 mph.	$h : r \rightarrow 5r + 120$
(3)(50) + 250	250 miles more than the distance a car goes in 3 hours at 50 mph.	$k : r \rightarrow 3r + 250$

The outputs of functions h and k can be described as "the total distance from the starting line to the finish line". The mathematical verb form implied here is equality, "=". Therefore we can write the following mathematical sentence:

$$5r + 120 = 3r + 250.$$

Exercises 18-3c

(Class Discussion)

Consider the following problem:

> "Two cars start from the same point at the same time and travel in the same direction at constant speeds of 34 and 45 miles per hour, respectively. In how many hours will they be 35 miles apart?"

Usually we let the variable represent a number that we are trying to find in the problem. If we do this in the above problem, then we can proceed as follows:

1. Let the variable h represent ___?___ .
2. The numbers 34 and 45 can specifically represent ___?___ .
3. The output of the function $f : h \to 45h$ can be interpreted as ___?___ .
4. The output of the function $g : h \to 34h$ can be interpreted as ___?___ .
5. The output of ~~the~~ function $k : h \to 45h - 34h$ can be interpreted as ___?___ .
6. The number 35 will represent ___?___ .
7. The mathematical verb form implied by the results of questions 5 and 6 is ___?___ .
8. The mathematical sentence which is a translation of the above problem is:

$$45h - 34h \;?\; \underline{\quad ? \quad}$$

Exercises 18-3d

Translate the following English sentences into mathematical sentences which could help solve the problems. You do not have to find the solution sets at this time. State clearly what each variable represents and be sure that you describe the outputs of each of the functions involved, as you develop your sentence, or sentences.

1. A first number is three more than a second number. The sum of the two numbers is 31. What are the numbers?

2. A car travels 232 miles in 4 hours. What is the average speed of the car?

3. An odd number which is increased by the next larger consecutive odd number is equal to 148. What are the two numbers?

4. The degree measure of the largest angle of a triangle is 20 more than twice the degree measure of the smallest angle. The degree measure of the third angle is 70. The sum of the measures of the angles of a triangle is 180. What is the measure of the smallest angle?

5. The area of a rectangular-shaped lot is 5000 square feet. The length of the lot is twice the width. Find the width of the lot. (Hint: Let w represent the number of feet in the width of a lot.)

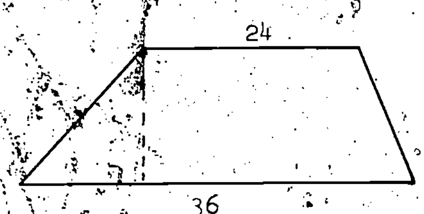
6. A boy runs against a head wind for one mile and averages 2 miles per hour. He returns one mile, running with the wind, and averages 5 miles per hour. How fast can the boy run in still air (no wind) and what is the speed of the wind? (Use two variables and write two mathematical sentences.)

7. Three consecutive odd integers add up to 75. What is the smallest of these three numbers?

8. Three consecutive odd integers add up to 57. What is the middle number of these three numbers?

9. Three consecutive odd integers add up to 117. What is the largest of these three numbers?

10. The area of a trapezoid is 512 square inches. The length of the bases are 24 inches and 36 inches respectively. What is the length of the altitude?



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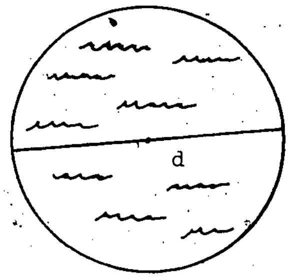
11. Horatio Algae built a circular swimming pool in his backyard. The pool covers an area of 1700 square feet. What is the diameter of the pool? (*Hint*: The area of a circle in terms of the radius of a circle, r, is given by the formula, $A = \pi r^2$.)



12. As a boy, Horatio Algae sold papers on a street corner. He was paid 1 cent for each paper sold during the week and 2 cents for each paper sold on Sunday. During one week he sold 1700 papers including Sunday sales. He received 22 dollars for the week. How many papers did he sell on Sunday? (Use two variables; write two open sentences.)

13. Horatio Lox, a cousin of Horatio Algae and a rocket fuel expert, was preparing a new mixture for an experimental missile. He found that the amount of liquid hydrogen must be exactly 17 percent of the rest of the secret ingredients. The total amount of the final mixture must be exactly 4 gallons. How much liquid hydrogen must be used?

In many problems you must deal with the process of converting one unit of measure to another. There are some parts of the translation procedure which could help you understand and complete the conversion from one unit to another.

Example 4: "The number of centimeters in one inch is 2.54. Find the length of a bar of silver, in centimeters, if the bar is one yard long."

(1) i represents a number of inches in a bar of silver.

(2) $f : i \rightarrow (2.54)(i)$ is a function whose output can be described as follows: "the number of centimeters in i inches".

(3) Since there are 36 inches in 1 yard, $36 \rightarrow (2.54)(36)$. Hence 91.44 is the value of the function for an input of 36.

Exercises 18-3e

For each of the following conversion problems state a function and find its output to answer the question.

1. The number of feet in a mile is 5280. How many miles does a runner go if he runs 11,320 feet?

2. The number of square inches in a square foot is 144. How many square feet are there in $7\frac{1}{2}$ square inches?

3. The number of minutes in one hour is 60. A car is traveling at 30 miles per hour. How many miles per minute is the car traveling?

4. The number of seconds in one minute is 60. A car is traveling at 30 mph. How many miles per second is the car traveling?

5. The number of feet in one mile is 5280. A car is traveling 30 miles per hour. What is the speed of the car in:

   (a) feet per hour?

   (b) feet per minute?

   (c) feet per second?

18-4. Using Drawings or Diagrams

It should be made clear to you that there are no universal procedures or formulas for analyzing and solving every problem. There are, however, some patterns of thought and some strategies which may help you to analyze and solve problems. Each problem situation is usually somewhat individual and it will probably require imaginative and creative thought to reach some satisfactory conclusions.

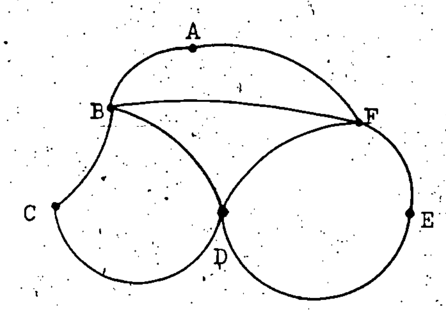
You have already developed some ability to translate or to create a certain kind of a mathematical model of a given problem situation. However, not all models have to be in the form of mathematical sentences. You could also use drawings, tables of data, graphs, or any combination of these and other methods which will help organize your analysis of the problem.

The following examples and exercises are illustrations of problem situations where a drawing, a sketch, or a geometric construction is a particularly helpful way of organizing the analysis of a problem. It is not possible to give a complete list of helpful diagrams, because each problem situation will usually demand a unique sketch to help analyze the situation. We will, however, discuss a variety of different diagrams to encourage you to use a version of this method as <u>you</u> develop <u>your</u> <u>own</u> problem analysis techniques.

<u>Exercises 18-4a</u>

(Class Discussion)

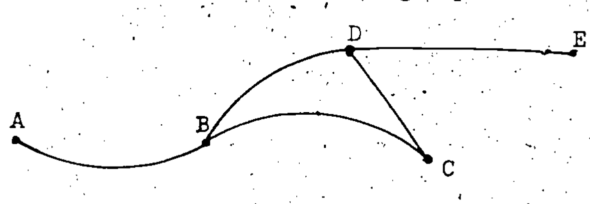
1. Look at the following map containing six grocery stores and nine streets connecting these stores. The Sheriff's patrol wants to find a route that will enable him to travel over each <u>street</u>, <u>exactly</u> <u>once</u>, as he checks the security of the stores and their connecting streets. The Milk Delivery man is interested in finding a route that will take him to each <u>store</u> <u>exactly</u> <u>once</u>, and he is not particularly interested in driving on each street.



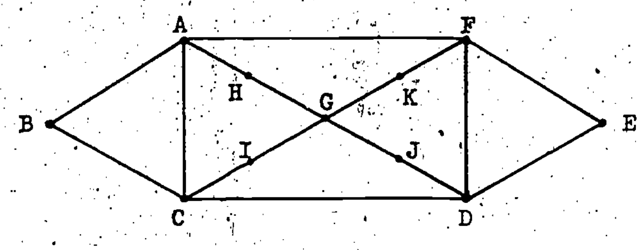
(a) Find a route for the Sheriff. (He doesn't mind going past the same store more than once.)

(b) Find a route for the Milk Delivery man. (He isn't particularly interested in traveling over each street.)

(c) Find a route for the Sheriff's patrol and a route for the Milk Delivery man on the following map.

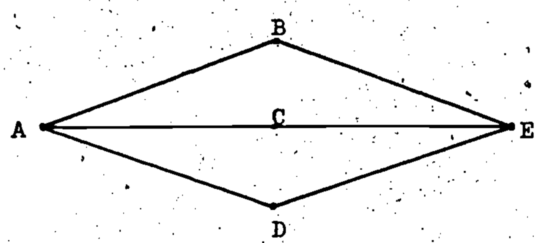


(d) Find a route for the Milk Delivery man and a route for the Sheriff's patrol on the following map.



2. Let's investigate some maps and some possible routes for the Sheriff's patrol.

(a) Suppose stores A and E have three streets connecting them with stores B, C, and D, respectively.



If you start at A and travel over each street exactly once, then where does the Sheriff end his patrol? Could the Sheriff start at B, or C, or D? Could he start at E?

(b) If a store has an odd number of streets connecting it with other stores, then would you conclude that the Sheriff must start or end his patrol at that store? Why or why not?

(c) If there are more than two stores that have an odd number of streets connecting them with other stores, then can the Sheriff find a patrol route? Why or why not?

(d) Draw another map of the streets connecting the five stores where no store has an odd number of streets connecting it to other stores. Is it possible to find a route for the Sheriff's patrol? If you start at A where must you end?

3. Suppose that the lengths of the sides of one equilateral triangle are double the lengths of the sides of a second equilateral triangle. How would you compare the area of the larger triangular region and the area of the smaller triangular region? (Make a guess now before you start.)

(a) Construct two equilateral triangles as described above.

(b) Find the midpoints of the sides of the larger triangle and connect these with segments.

(c) Now, how would you compare the area of the two original triangular regions? How does this compare with your guess?

(d) Can you justify your conclusion with a proof using SSS, SAS, or the formula for the area of a triangle? If so, do it.

Exercises 18-4b

In each of the following problems present the analysis of the information in the form of a diagram, a sketch, or a construction. It is not necessary to solve the problem unless you wish to do so. However, you should guess an answer, and indicate how the model you have drawn would help you analyze the problem situation.

1. A school is located five blocks east and six blocks north of the home of two brothers. The older brother walks four blocks east and two blocks north to his girl friend's house. They walk from there one block east to a donut shop, and then proceed directly to school. The younger brother cuts across vacant lots to a point one block directly west of school and then proceeds to school. How far does each boy walk to school?

2. An engineer was assigned the task of designing a method of building a "black box" for a secret space project. The box was to be constructed of a highly expensive metal alloy. The box was to be open at the top, and the faces were to be congruent square regions. Since the metal was so expensive, the pattern for each box had to be stamped out in one flat piece. He decided that he needed to know:

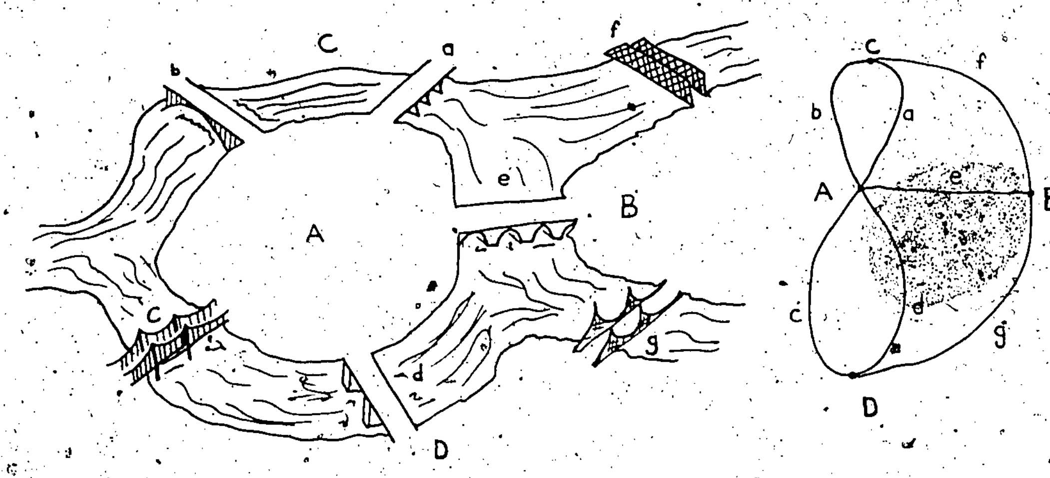
(a) How many different ways can five congruent squares be joined edge to edge?

(b) How many of the figures can be folded so as to form a box with an open top? Label the square region which is the bottom with the letter B.

(Hint: Draw diagrams representing 5 squares in a row; 4 squares in a row, but not 5; 3 squares in a row, but not 4; etc.)

3. For many thousands of years people have enjoyed working various kinds of problems. A good example of this is a problem concerning the Königsberg Bridges. In the early 1700's the city of Königsberg, Germany, was connected by seven bridges. Many people in the city at that time were interested in finding if it were possible to walk through the city so as to cross each bridge exactly once.

From the diagram showing these bridges, can you figure out whether or not this can be done? You may be interested in knowing that a mathematician answered this question in the year 1736.



4. A boy purchased 24 ft. of fencing to make a rectangular-shaped pen for his dog. What should the dimensions of the pen be so that the dog will have the largest play area? (Hint: Use graph paper to do the following.)

(a) Draw four different rectangles with a side of length 4, 6, 7, and 10 respectively and each with a perimeter of 24. Be sure to label the sides with their lengths.

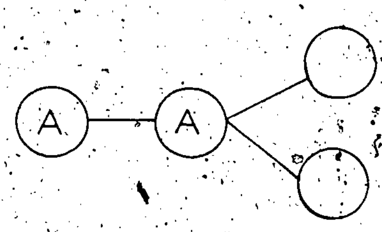
(b) Compare the areas of these figures by counting the squares in each rectangular region.

(c) Does there appear to be one rectangular region with larger area? You should be able to make a guess now which you will be able to prove in a later chapter.

5. If there are 3 roads from town X to town Y, and 4 roads from town Y to town Z, in how many ways may one drive from X to Z by way of Y? How many routes are possible for a round trip from X to Z and return?

6. Suppose the California Angels and the Los Angeles Dodgers are playing in the World Series, and that the Angels win the first two games, but lose the series. In how many different ways can this occur? (Copy and complete the following tree diagram.)

1st game 2nd game 3rd game 4th game 5th game 6th game 7th game



18-5. Organizing Information in Tabular Form

Many times it is especially useful to arrange information in tables. In fact, such an arrangement is very often the only efficient way of gaining any insight into the analysis of a problem situation. The following problems have been chosen because they particularly emphasize this phase of problem analysis. Remember that a single problem situation may be analyzed by various techniques. Be prepared to try several, discarding some along the way.

Exercises 18-5a

(Class Discussion)

(Exercises 1-3) There are three bus pickup points A, B, and C for taking students to school in the town of Shady Hills. A new school is to be built at one of two possible sites, X and Y. Point A is two miles from X and four miles from Y. B is three miles from X and three miles from Y. C is six miles from X and five miles from Y. 200 students are picked up at A, 250 students from B, and 225 students from C. It is desired to choose the site which will result in the minimum amount of travel to school by the town's student population. Which site is chosen?"

1. Draw a table like the following and fill in the missing entries.

From / To	A	B	C
X	2 miles		
Y			

Distance to X or Y from A, B, and C.

2. Draw a table like the following and fill in the missing entries.

From / To	A	B	C
X		750	
Y			

Total Student Distances

(These entries are found by multiplying the distance from each point to a site by the number of students picked up at that point.)

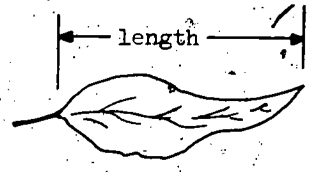
3. Which site, X or Y, was chosen? Explain how you arrived at your conclusion.

(Note: The student should recognize that cost is also minimized in the solution, and that this kind of a model can be used to represent a variety of other situations.)

(Exercises 4-6) "Is there an interesting relationship between the length and width of the leaves of a particular bush or tree?"

4. The class should select 20 to 40 leaves from a single tree or bush and measure the length and width of each leaf. Record these data in a table. Measure the width by creasing the leaf and measuring the

crease. To get the crease put the tip of the leaf on the point where the stem leaves the base of the leaf and fold it.



5. Add to this table, columns headed: the sum of the length and width, $L + W$; the difference, $L - W$; the product, LW; and the ratio, $\frac{L}{W}$. Calculate these values for each leaf.

6. Is there any particular relationship which seems to exist between the length and width of the leaves? If so, which column gave you your information?

Exercises 18-5b

1. The following list represents the results of 100 throws of a die, grouped in fives for convenience.

43553	53344	14166	53213	46451
54563	41353	35335	65536	64112
43253	62454	53263	33423	21531
24131	64235	26563	22522	21355

(a) Design a table which efficiently lists these data. Remember that the only outcomes are 1, 2, 3, 4, 5, and 6.

(b) Use your table to calculate the probability or rolling a 1; a 2; a 3; a 4; a 5; or a 6.

(c) How do the results of Exercise 2 compare with the theoretical probability of rolling a 1, 2, 3, 4, 5, or 6?

2. Consider the following sequence of sums of odd numbers: (1), (1 + 3), (1 + 3 + 5), (1 + 3 + 5 + 7), ...
(Remember that the symbol "..." means "and so on".)

(a) Make a table like the following and complete the entries.

Number of the term	1	2	3	4	5	6	7	8	...
Value of the term	1	4	9						

(b) If the number of the term is 25, then what is the value of the term? Did the table help you decide how to answer this question?

(c) If the value of the term is 144, how many odd numbers are added together to get that sum? What was the last odd number in the indicated sum for this term?

(d) If the number of the term is represented by n, then the value of the term could be represented by what?

3. Consider the following sequence of rational numbers.

$$\left(\frac{1}{1\cdot 2}\right),\ \left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}\right),\ \left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}\right),\ \left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}\right)\ \cdots$$

(a) Draw a table like the following and complete the entries.

Number of the term	1	2	3	4	5	6	7	8	...
Value of the term	$\frac{1}{2}$	$\frac{2}{3}$							

(b) Given that the number of the term is 21. What is the value of the term? Did your table help you decide how to answer this question?

(c) If the number of the term is represented by an integer n, can you represent the value of the term?

(d) If the value of the term is $\frac{13}{14}$, what is the number of the term? Write the fraction representing the last rational number in this term.

18-6. Estimation or Guessing

In this section we wish to focus your attention upon the technique of arbitrarily assigning real numbers to unknown quantities in a problem. This process has several advantages. By using specific numbers, you should be able: first, to use the many techniques of computation and simplification already known for the real numbers; second, to derive some useful information about "how good" your estimate is; and third, to gain some insight into how the more formal analysis could be organized. The ability to create and to interpret good numerical models of problem situations is an important skill to develop in problem analysis.

Exercises 18-6a

(Class Discussion)

(Exercises 1-7) Suppose that we stretch a wire around the earth at the equator. Assume that a smooth sphere of diameter 8000 miles can be used as a model of the earth. If we cut the wire, weld in another piece 100 feet long, then hold the wire above the surface so that it is the same distance above the earth all the way around, how far above the surface will the wire be? (Use $\pi \approx \frac{22}{7}$.)

1. Guess an answer and record it. Your first estimate is ___?___ ft. Let's see if it is too large or too small.

2. Represent the circumference of the earth in miles using 8000 as the diameter and $\frac{22}{7}$ for π. Write an indicated product which will represent the circumference in feet. (Remember there are 5280 feet in one mile.)

3. The diameter in feet of the new circle of wire will be $(8000)(5280) + (2) \times$ (your guess). Draw a figure and show why your estimate is multiplied by 2.

4. Write, as an indicated product, the circumference of the new wire circle, in feet.

5. Still using your estimate, how does the circumference of the new wire circle compare with the circumference of the original wire circle? Compare these two numbers by subtraction.

6. What should the difference be? Was your guess too large or too small?

7. Unless you see how to work the problem directly, revise your guess and check your results again.

(Exercises 8-13) Oliver Baconfat and his friend Shadow Jones race snails. The speed of Oliver's snail is 8 miles per week less than that of Shadow's snail. If Oliver's snail requires 5 weeks to go the same distance that Shadow's snail goes in $3\frac{2}{3}$ weeks, how fast does each snail travel in miles per week?

8. State clearly and concisely what each of the numbers 8, 5, and $3\frac{2}{3}$ represent.

9. Guess a value for the speed of Shadow's snail. Your guess must be greater than 8. Why?

10. Show how you would calculate the speed of Oliver's snail using your estimate for the speed of Shadow's snail.

11. Show how you would calculate the distance that Shadow's snail would travel in $3\frac{2}{3}$ weeks. Use your "guess" for the speed.

12. Show how you would calculate the distance that Oliver's snail travels in 5 weeks, using the result of Exercise 10.

13. Reread the problem. Do the snails travel the same distance? Are your values for Exercises 11 and 12 the same? Do you see a plan for finding a correct "guess" which will make them the same? If so, explain your plan. If not, try another "guess" for Exercise 9 and repeat Exercises 10-13.

Exercises 18-6b

1. My room has a doorway which is exactly 24 inches wide. My round fat uncle has a maximum circumference (by tape measure around the waist) of 72 inches. Can he get into the room? (Use $\pi \approx \frac{22}{7}$.)

   (a) Estimate the diameter of the "round uncle".

   (b) Check your estimate to see if you get a circumference of 72 inches. Was your estimate too large or too small?

   (c) Explain how to solve this problem.

2. The radar operator on an aircraft carrier detects a contact moving directly toward the carrier. He estimates the distance to the contact at 1400 miles and the speed of the contact at 850 miles per hour. How long will it take one of the carrier's planes to intercept the contact if it flies directly toward the contact at 950 miles per hour?

   (a) Guess the time, in hours, it takes for the carrier's plane to intercept the contact. ___?___ hours.

   (b) How far does the contact fly in the time that you estimated?

   (c) How far does the aircraft carrier plane fly in the time you guessed?

(d) If you add the two distances you calculated in Exercises 5 and 6, then what should that sum represent?

(e) Explain how you can use the method you used in checking your estimate to solve the problem. (Write the mathematical sentence involved.)

3. The edges of two cubes differ by 2 inches, and their volumes differ by 728 cubic inches. Find the dimensions of each cube.

(a) Estimate the length of the edge of the larger cube. It must be greater than or equal to 9. Why?

(b) Calculate the length of the edge of the smaller cube using your estimate from Exercise 9. (Show clearly how your calculation is done and label the result.)

(c) Calculate the volumes of the cubes using your estimate and the result from Exercise 10.

(d) Is the difference between the volumes 728? Was your estimate too large or too small?

(e) If you can see how to write mathematical sentences that will serve as models of this situation, then do so. If not, revise your estimate and repeat Exercises 9-12.

4. A scientist has designed a square solar battery with a side of 3 centimeters. This solar battery is just large enough, in area, to provide enough power to operate one camera in a satellite. He now wishes to construct a battery which is exactly double the area of the first solar battery cell. What should be the length of its side?

(a) Calculate the area of the original solar cell. Double this area to get the area of the new cell.

(b) Estimate the length of the side of the new solar cell. Using your estimate, calculate the area of the new cell.

(c) How did the result of (a) compare with the calculated value of the area in (b)? (Too large or too small?)

(d) Revise your estimate of the length of the side of the new cell and compare again. Show your work.

(e) Do you think that you can find an exact decimal value for the length of the side of the new cell? If so, show how.

(f) Do you have any idea now about how long the side should be? Between what two integers does the correct value seem to be?

(g) Write a mathematical sentence which could serve as a model for this problem situation.

18-7. Solving Simpler Problems

Genuine problems, in their original form, are almost always too complicated to understand or to solve. Such problems can often be solved or analyzed by removing some of the complications and by constructing a mathematical model of a simpler, but similar situation. One has to be careful about conclusions found by such a method, but it would be foolish to reject possible conclusions that might be confirmed by experience or by more rigorous reasoning.

Exercises 18-7a

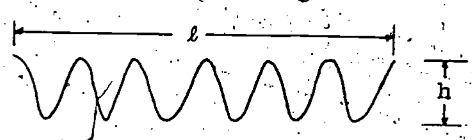
(Class Discussion)

"What is the longest line segment that can be drawn in the interior of a sphere from a given point on the sphere to a different point on the sphere?"

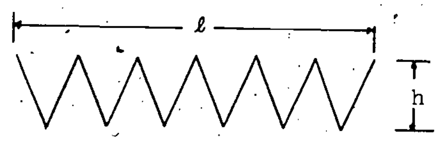
1. In a plane, what is a simpler figure that would be related to a sphere? Draw the figure.
2. Now let's pick an arbitrary point A on the circle (hopefully you chose a circle in Exercise 1), and draw a diameter through center C, to point B on the circle. Do you think this is the longest line segment that can be drawn in the interior of the circle?
3. Draw any other line segment in the interior of the circle from A to a different point D on the circle.
4. What is true about $\overline{AC}$, $\overline{BC}$, $\overline{DC}$? $(AC + CB)$ and $(AC + DC)$?
5. What is true about $(AC + DC)$ and AD?
6. Can you make a statement about a diameter $\overline{AB}$ and any other line segment drawn in the interior of the circle from point A?
7. Discuss how you can extend this analysis to a sphere.

Exercises 18-7b.

1. In a certain clothmaking mill a very high speed camera took various pictures of the fibers as they sped through the mill machinery and each looked something like the following:



The problem was to determine how long the fibers would be if they were stretched out straight. It was necessary to know the length in order to adjust and operate the weaving machines. The mathematician called in to advise the mill suggested solving this problem. Is it a simpler problem?



If ℓ is 42 inches and h is .3 inches, how long was the approximate length of the fiber?

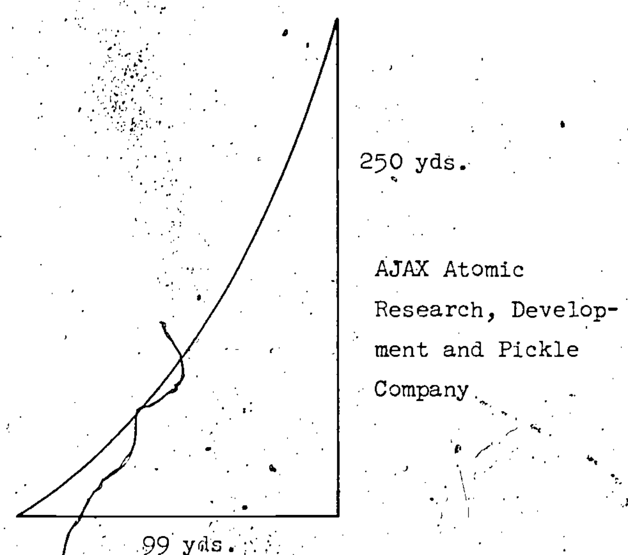
2. It is desired to calculate the approximate volume of a man given the average thickness of his head, arms, trunk, and legs. Design a model of a man and calculate the volume of this man given that the average thickness of the head is 7 inches, his arms 3 inches, his trunk 10 inches, and his legs 4 inches. Select appropriate dimensions of your own for the lengths of each of the parts mentioned above.

3. Often lumbermen need to estimate how much lumber, measured as volume of wood, they can get from a stand of trees. They can usually get for each stand of trees marked for cutting, an average height and an average circumference, measured at the base of the trees. What kind of a model would you recommend to the lumberman? If the average height is 80 feet and the average circumference was 20 feet, how many cubic feet of lumber would there be in one tree, using your suggested model?

4. Lumber mills also need to know the volume of wood in the logs they process. Propose a good model for this problem. If the average diameter of the small end of the logs is 24 inches and the average

length is 40 ft., how many cubic inches of wood would there be in an average log? Usually lumber is measured in "board feet", the volume of lumber in a piece of wood in the form of a rectangular solid 1 in. by 12 in. by 12 in. or 144 cubic inches. Would this mean that it would be best to choose a model of a log that was a rectangular solid? How many board feet are there in an average log mentioned above?

5. The new lawn of a large research corporation was bounded on the front by a curved street as in the following diagram. The company decided to establish the lawn immediately by having sod placed instead of planting seed. The sod comes in long rectangular strips. To order the proper amount of sod they needed to know the area of the lawn. If the sod comes in strips 3 ft. wide and 40 ft. long and the dimensions of the lawn are as given in the drawing, recommend a model which will solve their problem.



18-8. Using Basic Properties

The next exercises illustrate an important phase of analysis in mathematics. This is the analysis of the structure of any system for basic properties like "closure", "commutativity", and the like.

You have already used this process, rather informally, as you have studied and applied your mathematics. For example: In the study of the measure of area, by intuitively establishing a functional relationship between the set of plane geometric regions and the set of positive real numbers, you immediately have all of the techniques and ideas developed for positive real numbers available to help you think about areas of plane geometric regions. In other words, the areas of plane geometric figures have the same basic properties as the positive real numbers, and hence you can interpret areas as positive real numbers whenever convenient. This process of using a known system with a known structure to investigate other analogous systems is the basis for the early development of algebra. It can be shown that the set of algebraic expressions has exactly the same structure as the set of real numbers, and hence all of the techniques, theorems, and the like, from the arithmetic of real numbers are available for use in algebra.

Now consider the following situation.

Imagine that, in the near future, you land with a space expedition on Modulus Three (the third planet of the star Modulus), and find a civilization which is friendly and able to communicate with you. On this planet the need for measurement of land has never arisen. Hence the need for numbers may not have occurred except for one peculiar circumstance.

This planet has a single moon which has a period of rotation of exactly (what we would call) three days. This has taken on mystical significance which has in turn led to the use of numbers as a record of the Moon's position. They use the pecular number names mon, tu, wed, for these phases where we would use the words one, two, three. Since the phases of the moon are cyclic, their entire counting system runs mon, tu, wed, mon, tu, wed, mon, tu, wed, etc., as 1, 2, 3, 1, 2, 3, 1, 2, 3, etc.

Recently the need for operations on these numbers has led to the development of arithmetic addition and multiplication along the same lines as they did for us except that, due to the mystical background, the inventions of any compounds through the use of "base" or "place value" has been strictly forbidden. The Modulians have now recognized the need for an algebra for their number system and you have been asked to help. It will now be your job to analyze their number system and develop a primitive algebra necessary for the solution of some simple equations in their system.

The tables for the arithmetic facts developed so far are as follows:

+	1	2	3
1	2	3	1
2	3	1	2
3	1	2	3

$\cdot$	1	2	3
1	1	2	3
2	2	1	3
3	3	3	3

Exercises 18-8a

(Class Discussion)

Using the qualities which are either evident from the tables or are suggested by the exercises, complete your analysis of the Basic Properties of this system.

1. Under "+" does the system have:

   (a) Closure?

   (b) Commutativity?

   (c) Associativity? Is

   (1) $(1 + 2) + 3 \stackrel{?}{=} 1 + (2 + 3)$

   (2) $(1 + 2) + 2 \stackrel{?}{=} 1 + (2 + 2)$

   (3) $(1 + 3) + 2 \stackrel{?}{=} 1 + (3 + 2)$

   (4) $(2 + 3) + 2 \stackrel{?}{=} 2 + (3 + 2)$

   (d) An identity element? If so, what is it?

   (e) An additive inverse for each element? Using $^{\sim}a$ to represent the inverse such that $a + {}^{\sim}a$ = the additive identity, which of the additive inverses exist?

   $$^{\sim}1 = ?, \quad ^{\sim}2 = ?, \quad ^{\sim}3 = ?.$$

2. Under "$\cdot$" does the system have:

   (a) Closure?

   (b) Commutativity?

   (c) Associativity? Is

   (1) $(1 \cdot 2) \cdot 3 \stackrel{?}{=} 1 \cdot (2 \cdot 3)$

(2) $(1\cdot2)\cdot 2 \stackrel{?}{=} 1\cdot(2\cdot2)$

(3) $(1\cdot3)\cdot 2 \stackrel{?}{=} 1\cdot(3\cdot2)$

(4) $(2\cdot3)\cdot 2 \stackrel{?}{=} 2\cdot(3\cdot2)$

(d) An identity element? If so, what is it?

(e) A multiplicative inverse for each element? If we use $1^{\sim}$, $2^{\sim}$, $3^{\sim}$ to represent the inverses, show which of the multiplicative inverses exist.

$1^{\sim} = ?$, $2^{\sim} = ?$, $3^{\sim} = ?$ (In other words, for each number, a, in the system is there a unique number $a^{\sim}$ such that $a \cdot a^{\sim}$ = multiplicative identity ?)

3. Is "$\cdot$" distributive over "+" from the right? From the left?

(a) $3\cdot(2+1) \stackrel{?}{=} 3\cdot2 + 3\cdot2$

(b) $2(1+3) \stackrel{?}{=} 2\cdot1 + 2\cdot3$

(c) $2(2+3) \stackrel{?}{=} 2\cdot2 + 2\cdot3$

(d) $(2+1)\cdot 3 \stackrel{?}{=} 2\cdot3 + 1\cdot3$

(e) $(1+3)\cdot 2 \stackrel{?}{=} 1\cdot2 + 3\cdot2$

(f) $(2+3)\cdot 2 \stackrel{?}{=} 2\cdot2 + 3\cdot2$

4. Does the structure of this system seem to be the same as the structure of the rational number system?

If this system has all of the above properties, then we can use the techniques we have developed for the rational number system to create equivalent equations in simpler and simpler form until the solution set becomes obvious.

Example: Find the solution set of $2\cdot x + 1 = 3$ in the Modulus Three System and explain your steps in terms of the basic properties of the system.

$2\cdot x + 1 = 3$

$2\cdot x + (1 + 2) = (3 + 2)$ — Adding 2 to both sides of the equation. Two was chosen because 2 is the additive inverse of 1.

$2\cdot x + 3 = 2$ — A number plus its inverse is the additive identity.

$2\cdot x = 2$ — If 3 is added to any number the result remains the same since 3 is the additive identity for this system.

$2 \cdot 2 \cdot x$	$= 2 \cdot 2$	Multiply both sides of the equation by 2. Two was chosen because it is the multiplicative inverse of 2.
x	$= 1$	

Solution Set: {1}.

Of course, some of these steps can be combined or shortened, but remember, it must be clear to someone else why you have solved the equation the way you did.

Exercises 18-8b

Solve the following equations in the Modulus Three System and explain your work so that the Modulians could see that the basic properties of their number system were being used.

1. $x + 2 = 1$
2. $x + 3 = x$
3. $x + 2 = x$
4. $3 \cdot x + 1 = 2$
5. $(2x + 1) + (x + 3) = 2$

18-9. Using Reasoning Techniques

Mathematics is a way of thinking, a way of reasoning. Some of mathematics involves experimentation and observation, but most of mathematics is concerned with deductive reasoning.

By deductive reasoning we show that from certain given conditions, a definite conclusion necessarily follows. Mathematicians use reasoning of this sort. They prove "if - - then" statements. That is, by logical reasoning they prove that if something is true, then something else must be true. You can often find all the different ways in which a problem can be solved and sometimes you can show by reasoning that a problem has no solution. The following example provides a simple illustration of a reasoning process.

Example: Suppose that there are thirty pupils in your classroom. Prove that there are <u>at least</u> two of them who have birthdays during the same month.

(1) Imagine 12 boxes with the names of the months on them.

(2) Imagine that each student's name is written on a slip of paper along with the month of his birthday.

(3) If only one slip of paper is put in each box, then there could <u>not</u> be more than 12 names in all.

(4) Since there are 30 names, then <u>at least</u> one box must contain more than one name.

Exercises 18-9a

(Class Discussion)

1. By a similar reasoning process, show that at <u>least</u> three members of the class of 30 students must have birthdays in the same month.

2. How many students would there have to be in order to guarantee that at least 4 students would have birthdays in the same month?

3. The following problem will illustrate the very helpful process of indirect reasoning.

   If a, b, c are positive integers, if a is a factor of b, and if a is <u>not</u> a factor of c, then a is <u>not</u> a factor of $(b + c)$.

   Let's investigate this statement using $a = 2$.

   (a) 2 is a factor of 32 and 2 is <u>not</u> a factor of 27. Is 2 a factor of $(32 + 27)$? Do you think the above statement about a, b, and c is true or false? Let's continue and justify your conclusion.

   (b) If 2 is a factor of a positive integer b, then this means that $b = (?)p$, where p is a positive integer.

   (c) If 2 is <u>not</u> a factor of the positive integer, c, then this means that $c \neq 2(?)$.

   (d) Let's <u>assume</u> that 2 <u>is</u> a factor of $(b + c)$. This means that $b + c = (?)(q)$, where q is some positive integer. If this assumption is false, then it should lead to a contradiction of a known fact about a, b, or c.

(e) We now have

$$b = 2p,$$

and

$$b + c = 2q.$$ Replacing b by $2p$, we get

$$2p + c = 2q.$$ Subtracting $(2p)$ from both sides, we get

$$c = 2q - 2p.$$ Simplifying by use of the distributive property we get

$$c = 2(q - p).$$

This last sentence means that 2 is a ___?___ of c, since $(q - p)$ is a positive integer.

(f) Since the above statement contradicts the known fact that 2 is not a factor of c, we can conclude that our assumption, part (d), is false. Hence the correct conclusion is that 2 is ___?___ a factor of $(b + c)$. (Of course, a completely general proof can be made by replacing 2 by a, where a is a positive integer.)

Exercises 18-9b

Show clearly the reasoning process necessary to solve the following problems.

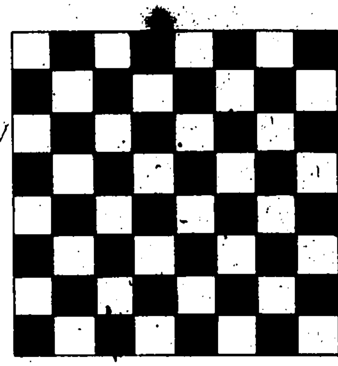
1. What is the least number of students that could be enrolled in a school so that you could be sure that there are at least two students with the same birthday? (same day, and same month)

2. What is the largest number of students that could be enrolled in a school so that you could be sure that they all have different birthdays?

3. In a class of 32 students various committees are to be formed. No student can be on more than one committee. Each committee contains from 5 to 8 students. What is the largest number of committees that can be formed?

4. What is the answer to Exercise 3 if every student can be on either one or two committees?

5. A checkerboard has 64 squares. Suppose that you have 32 cards and each card covers two checkerboard squares exactly. If you cut off two squares, one at each end of a diagonal of the checkerboard, and throw away one card, then is it possible to place the 31 remaining cards on the board so that the remaining 62 checkerboard squares are covered? Show how to do it, or prove that it is impossible.



6. Suppose that you select three cards from a deck, two black cards and one red card. A game is played by two people. The cards are shuffled and one card is placed face down in front of each player and one between them. Each player then looks at his card and the first one to identify the color of his opponent's card wins the game. Explain the reasoning used in deciding the best method of playing the game.

7. You remember that a prime number is a positive whole number, greater than 1, which cannot be written as the product of two smaller factors. The first few primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, ... It's easy to see that this sequence does not follow any simple law. In fact, the structure of the sequence turns out to be extremely complicated. However, the question, "Does the sequence of primes eventually end?" can be answered.

(a) $(2 \cdot 3) + 1 = ?$ Is 2, or 3 a factor of the result? Why or why not?

(b) $(2 \cdot 3 \cdot 5) + 1 = ?$ Is 2, 3, or 5 a factor of the result? Why or why not?

(c) $(2 \cdot 3 \cdot 5 \cdot 7) + 1 = ?$ Is 2, 3, 5, or 7 a factor of the result? Why or why not?

(d) $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) + 1 = ?$ Is 2, 3, 5, 7 or 11 a factor of the result? Why or why not?

(e) Let $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \ldots p) + 1 = N$, where p is supposed to be the largest prime in the sequence of primes from 2 to p. Will any of the primes in this sequence be a factor of N? Why or why not?

(f) Tell why N is a prime greater than p, or has a prime factor greater than p. In either case a prime greater than p has been found. This means that no matter how large p is there is always a prime greater than p. In other words, the sequence of primes continues indefinitely.

18-10. Summary

There are several overlapping problem analysis techniques which we have studied in this chapter. They are:

(1) Translation: Assigning convenient and meaningful symbols to represent various parts of the problem.

(2) Organizing information in tables: Arranging data in an organized form and deriving additional information from the table.

(3) Picturing relationships: Organizing information through geometric representations of the problem situation.

(4) Analogy: Comparing problem situations with other solved or understood problems that have the same or some similar characteristics or looking for basic structural properties in the problem situation.

(5) Using functions: Looking for the underlying functional relationships in the problem situation and clearly identifying these relationships. This technique was not treated separately, but was evident in most of the sections.

(6) Estimation: Assigning [illegible] a given problem situation to take advantage of the [illegible]es and techniques already developed for the real num[illegible].

(7) Reasoning: Using established [illegible] of logical reasoning to determine additional specific information about the problem situation.

Of course this list does not include all of the possible analysis techniques, but it will serve as a basic list which should be refined and expanded as you develop your own problem solving strategies.

In addition to the specific techniques for problem analysis studied in this chapter, you should also develop some good attitudes about problem analysis.

(1) Be willing to spend some time just thinking about and experimenting with a problem situation.

(2) Learn to be flexible in your approach. Be willing to try the unusual or bizarre approach in some situations.

(3) Be willing to improve your techniques for communicating your mathematical ideas to yourself and to others.

(4) Don't be discouraged if all problems are not solved or analyzed in the same way. Worthwhile problem situations often times occur in unusual circumstances.

Problem analysis is one of the major activities which concerns all of those who study, learn, and use mathematics. Problems are solved in mathematics both to extend and develop new and important parts of mathematics, and to help solve and interpret problems that occur in many different situations, including some that appear to be quite unrelated to the field of mathematics.

Some important techniques for problem analysis have been illustrated in this chapter. You should realize by now that these techniques will probably not be used in isolation, but that an approach should be developed, by you, which will use a collection of these skills to help you analyze and solve problems.